# LIGHT
# FOR STUDENTS

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# LIGHT

## FOR STUDENTS

BY

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# PREFACE

THIS book has been written to meet the requirements of students who wish to obtain an accurate and comprehensive knowledge of Geometrical and Physical Optics. In many instances results of recent researches are described, in connection with important laws which they elucidate. The mathematical investigations have, in all cases, been rendered as simple as possible, and have been developed so as to direct attention to the physical aspect of the subject. No knowledge of the Calculus is assumed on the part of the student. A number of illustrative experiments which may readily be performed are fully described, and numerous questions, mostly selected from public examination papers, are appended.

The first ten chapters are devoted to Geometrical Optics, *i.e.*, to explaining the consequences of the laws of Reflection and Refraction of Light. Some of the most important optical instruments, including the eye, are dealt with. Points which commonly present difficulties to students, such, for example, as the method of achromatising an eye-piece, are explained in considerable detail.

The remaining ten chapters are devoted to the development of the Wave Theory of Light. While ample attention is devoted to the more elementary parts of the subject, full explanations are also given of many points not usually dealt with in books of similar scope; as instances, the investigation of the velocity of transverse waves in an elastic solid, and Sellmeier's Theory of Dispersion, may be mentioned. The importance of the results

obtained will, I believe, justify the inclusion of these researches, the more so as the reasoning used can be easily followed by the average student. I regret greatly that the limited space at my disposal has prevented me from including a simplified account of that most beautiful and fertile of all optical theories—Maxwell's Electro-Magnetic Theory of Light.

Of the 306 figures used to illustrate the text, most of the line diagrams have been reproduced from original drawings. My indebtedness to original memoirs is acknowledged in the text; I must, however, here return my thanks to Mr. W. B. Croft, M.A., who has placed his valuable collection of Diffraction and Polarisation photographs at my disposal; and to Mr. C. P. Butler, A.R.C.Sc., who has provided me with several interesting spectrum photographs. Finally, my best thanks are due to Sir Richard Gregory, and Mr. A. T. Simmons, B.Sc., for their courteous assistance and advice while the sheets have been passing through the press.

EDWIN EDSER.

*September*, 1902.

# CONTENTS

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## CHAPTER I

### FUNDAMENTAL PROPERTIES OF LIGHT

**Introductory.**—Our knowledge of the external world is derived, primarily, from the mental examination and comparison of sense impressions. Our most trustworthy impressions are obtained, through the sense of touch, from the actual contact of external objects with parts of the human body. Scarcely less important to us, though more frequently vitiated by illusions, are the impressions obtained through the visual sense, or sense of sight. In this case there is no obvious connecting link between the object seen and the person who sees it. On covering our eyes we can no longer see anything, so that the eye is obviously the organ of sight. But in what manner can a distant object affect the eye so as to produce a visual impression? This question has occupied the minds of many of the greatest thinkers since the earliest times recorded in history; it has been answered, in a satisfactory manner, only during the last century, and even at the present day there are points which require explanation. But from the earliest times the need has been felt of postulating some agency by means of which the object seen influences the eye which sees it; this agency is termed **light.** Thus we do not directly *observe* the existence of light, but *infer* this in order to explain the formation of visual impressions. Consequently, the statement sometimes made, that we do not see objects, but the light which proceeds from them to the eye, is inaccurate.

The ancients supposed light to be something which proceeded from the eye of the observer to the object seen. If we try to attach any definite meaning to such a supposition, we must think of light as resembling tentacles stretching from the eye to surrounding objects. Such a theory has scarcely anything in its favour. It is more logical and more natural to think of light as something which proceeds from the object seen and affects the eye which it reaches. The nature and properties of this "something" will occupy our attention in the ensuing pages.

As we infer that visual sense impressions are produced by an agency called light, so we infer that the total absence of visual sense impressions, at any rate when the eye is healthy, is due to the absence of light from the eye. In other words, the condition which we term darkness is due to the absence of light. In a room, the windows of which are carefully closed by shutters, certain objects—such as a candle-flame, a glow-worm, or a patch of luminous paint which has been exposed to sunlight—will be more or less visible; such objects are said to emit light, or to be **self-luminous**. A candle-flame will not only itself be visible when introduced into a dark room, but will render the walls and furniture of the room visible also. Since, in the absence of the candle, the walls and furniture of the room could not be seen, the latter are not self-luminous; when seen, they are rendered visible by light derived from some self-luminous body. Thus, we infer that light from the candle-flame not only reaches the eye directly, but some of it falls on the walls of the room, and is thence thrown back so as to reach the eye.

**Rectilinear Path of Light.**—A minute object, held between the eye and a very small source of light, renders the latter invisible. It therefore intercepts the light from the source which would otherwise have reached the eye. From this we infer that light does not appreciably bend round an obstacle; or, in other words, **light travels in straight lines**. As we shall see, this is only true when the path of the light is in a uniform medium; when light passes from one medium to another (as, for instance, from air to water), the light which enters the second medium does not generally travel along a continuation of the straight line which formed its previous path. Further, under certain conditions, light does *to a very small extent* bend round an obstacle; the results of this bending will be fully

considered in a subsequent chapter, but need not concern us at present.

The term **ray** is applied to the rectilinear path along which light travels, in any direction, from a point in a luminous object. If the object emits light in all directions, then any straight line from a point of the object constitutes a ray. A collection of rays, proceeding from or toward a point is termed a **pencil**.[1] Thus each point of a luminous object gives rise to a number of pencils of light. When the light proceeds *from* a point, the pencil is termed divergent ; when *toward* a point, convergent. When a pencil diverges from, or converges toward, a point at a great distance from the observer, the component rays will be approximately parallel, and the approximation to parallelism increases with the distance of the point from the observer. We may say that **rays converging toward, or diverging from, a point at an infinite distance, form a parallel pencil.** As an instance, light rays, reaching the earth from a star, are sensibly parallel.

A collection of rays, proceeding from various parts of a luminous object, is termed a **beam of light**. Thus, sunlight, when admitted into a darkened room through a small orifice, forms a sunbeam. The path of a sunbeam in the air is often made visible by the light thrown off, or scattered, from small particles of floating dust, &c. The beam itself is, of course, invisible.

**Shadows.**—Let S (Fig. 1) be a small source of light, approximating to a geometrical point, whilst K is an obstacle which intercepts the light which falls on it. Since the light rays which fall on K are intercepted, whilst those which just pass it are not appreciably bent or modified, it follows

FIG. 1.—Formation of a Shadow Cone.

[1] Latin, *penicillum*, a painter's brush. The similarity between light rays converging toward a point, and the converging hairs of a pointed brush, is sufficiently obvious.

that a shadow cone extends away from K, and a point within this cone receives no light from S. A screen held at right angles to the axis of the shadow cone will show a well-defined shadow of the obstacle. Very sharp shadows of objects are thrown by the light from a naked arc lamp.

When the source of light is large in comparison with its distance from the obstacle, the light from each point of the source throws a separate shadow cone from the obstacle, and it is only the space common to all of these shadow cones which is free from light. Fig. 2 gives the sections of the shadow cones thrown from a sphere, B, by the light from opposite points of an extended source, A. The conical space BS receives no light from any part of A; any point in the shaded portion of the figure receives light from some parts of A but not from others. A screen, *mn*, placed between B and S, will show a perfectly black central portion, called the **umbra**, surrounded by an area partially in shadow, called the **penumbra**. If the screen is placed beyond the apex S of the umbral cone, the shadow will only show a penumbra. A shadow thrown by an ordinary lamp or gas flame generally shows a penumbra, with or without an umbra. The light from the sun throws umbral and penumbral cones from the moon, and when a point on the earth passes into either of these, an eclipse of the sun occurs. When the point on the earth is within the penumbral cone, the eclipse is *partial*, and part of the sun is seen. When the point on the earth passes into the umbral cone, the eclipse is *total*, and the whole of the sun is obscured.

FIG. 2.—Formation of Umbral and Penumbral Cones.

The sun also throws a shadow cone from the earth, and when the moon moves into this it becomes eclipsed. As the earth rotates, a point on its surface is exposed to the light from the sun during the day, and withdrawn into the shadow cone at the advent of night.

**The Pin-hole Camera.**—Let AB (Fig. 3) represent a luminous object, placed in front of a small aperture, C, pierced in one side of an otherwise closed chamber. Since the light rays diverging from points of AB are rectilinear, a cone of rays of small vertical angle will pass through C from each point of AB. Let the side of the chamber remote from C be covered by a white screen. Then each cone of rays illuminates a small spot on the screen, and if the aperture C is very small, each of these illuminated spots approximates, in dimensions, to a point. Thus, for each point of AB there will be a corresponding bright point on the screen; in other words, a luminous image, A′B′, corresponding to the object AB, will be formed on the screen. From Fig. 3 it is readily seen that the image differs from the object in being inverted. Further, if DC $= u$, whilst CE $= v$, then—

FIG. 3.—Pin-hole Camera.

$$A'B'/AB = v/u.$$

Thus, the image can be made as large as we please by increasing $v$, or decreasing $u$. As we decrease $u$, the pencil of rays from a point on the object becomes more divergent, so that a larger area on the screen is illuminated by the light from a particular point of the object. This produces a blurring of the image. It would at first sight appear that increasing $v$ should also increase the blurring of the image; but, as a matter of fact, the definition of the image *increases* with $v$, up to a certain point, and subsequently decreases. Decreasing the size of the aperture increases the definition up to a certain point; but if the aperture be diminished beyond a certain magnitude, depending on the values of $u$ and $v$, the definition of the image decreases. These results, as we shall find, admit of a ready explanation in terms of the Wave Theory of Light.

The *form* of the image is independent of the shape of the aperture. The latter merely influences the shape of the individual bright spots on the screen, corresponding to different points on the luminous object. When sunlight falls through the interstices between the leaves of a dense

forest, it paints oval bright spots on the ground. Each bright spot is an image of the sun, and would be circular if the sun were exactly overhead.

If the image A′B′ is formed on the sensitive film of a photographic plate, a permanent image will be formed on the plate after development. The use of a pin-hole camera strongly recommends itself for the photography of buildings, since the image is an exact facsimile of the object, while a lens, unless this is specially designed for the photography of buildings, generally produces a distorted image. The only disadvantage of the pin-hole camera lies in the protracted exposure required.

**Transparency and Opacity.**—Material substances may be broadly divided into two classes: those through which light can pass, which are termed **transparent**; and those which intercept the light which falls on them, which are termed **opaque**. Some substances are penetrable, to a greater or less extent, by light, but an object cannot be seen distinctly through them; such substances are said to be **translucent**. A fog, paraffin wax, and a weak solution of milk and water, are translucent to light. A transparent substance may be rendered translucent by roughening its surface. Ground glass forms a familiar instance of this transformation.

No substance is either absolutely transparent, or absolutely opaque. Air, water, and glass intercept some light, while thin layers of metal transmit a certain amount of light.

**Reflection.**—When a narrow pencil of light falls on a smooth polished surface, another pencil, termed the **reflected pencil**, is thrown off from the point of incidence. The laws of reflection have been known from a very remote period, and may be stated as follows. Let IC (Fig. 4) be a ray of light incident on a reflecting surface at C. Draw the straight line CN perpendicular to the reflecting surface at C; this line is termed the **normal** to the surface at C. Then, **the reflected ray** CR **lies in the plane containing the incident ray** IC **and the normal to the surface at** C.

FIG. 4.—Incident and Reflected Rays.

Further, **the incident and reflected rays are equally inclined to**

**the normal, and lie on opposite sides of it.** The **angle of incidence**, $i$, is the angle NCI. The **angle of reflection**, $r$, is the angle NCR. From the above law—

$$i = r.$$

The above laws apply to the reflection of light from any perfectly smooth surface, whatever may be its form, or the nature of the medium which it bounds. If the surface is curved, we may divide it into very small elements of area (which will be approximately plane), and draw the normals to these elements. Light will then be reflected from each element according to the above law.

Some substances, such as polished silver, reflect nearly all (*i.e.* more than 90 per cent.) of the light which falls on them. These substances are said to be good reflectors. Light is also reflected, though in smaller proportion, from the smooth surfaces of transparent media. About 4 per cent. of the light, incident at a small angle on a glass surface, is reflected. When light is incident similarly on the surface of water about 1·7 per cent. of it is reflected.

When light is incident on a rough surface (such as that of ground glass), the above laws of reflection do not hold good. A narrow parallel pencil of light does not give rise to a parallel pencil, but to a system of rays diverging from the point of incidence. The reflection in this case is said to be **diffusive.** The surface of a piece of white unglazed paper reflects light diffusively. A certain amount of light is reflected regularly (*i.e.* according to the laws explained above) from the surface of white, *glazed* paper.

**Refraction.**—When a ray of light is incident on the smooth surface of a transparent medium, a reflected ray is not alone formed. A second ray starts from the point of incidence and traverses the transparent medium; this is termed the **refracted ray.** The direction of the refracted ray does not, in general, agree with that of the incident ray, but there is a definite relation between these two directions.

Fig. 5.—Incident and Refracted Rays.

Let IC (Fig. 5) be the incident ray, lying in the plane of the paper. Let this ray meet the surface which separates two

different media at C. If the surface in question is perpendicular to the plane of the paper, the normal CN to the surface at C will lie in the plane of the paper. Produce NC to N′. Let CR be the refracted ray. Then, **the refracted ray CR lies in the plane containing the incident ray IC and the normal NCN′. The angle of incidence $i$ ($= \angle$ NCI) and the angle of refraction $r$ ($= \angle$ N′CR) are connected by the relation—**

$$\sin i/\sin r = \text{a constant} = \mu \text{ (say)}.$$

This relation is generally termed **Snell's Law**, from its discoverer, Willebrod Snellius. Snell's law applies to any case where a ray of light meets the interface between two different media, provided these media possess the same properties in all directions. It therefore applies to refraction at the interface between air and glass, air and water, water and glass, &c. When one of the media is crystalline, or when both are crystalline, the law of refraction becomes more complicated; consideration of these cases may be postponed for the present. The constant $\mu$ is termed the **index of refraction**; its value depends on the nature of the two media separated by the refracting surface. When the light, before meeting the surface, has been travelling in a vacuum, the constant $\mu$ is termed the refractive index of the transparent medium in which the refracted ray is formed. When light is incident in air on a transparent medium, the index of refraction is practically the same as if the light had been incident in a vacuum on the same surface.

When $\mu$ is greater than unity, it follows that $r$ is less than $i$, and the refracted ray makes an angle with the normal which is smaller than the angle of incidence. Thus, considering the incident and refracted rays as parts of a single ray, we may say that when $\mu > 1$, the ray is deflected at the point of incidence, being bent toward the normal. This deflection generally occurs when light passes from a rarer to a denser medium; and when the light is deflected at the surface of separation of two media, so as to be bent toward the normal, we say that the second medium is **optically denser** than the first, whatever the mechanical densities of the media may be. Conversely, when light is deflected at the surface of separation of two media, so as to be bent away from the normal, the index of refraction, $\mu$, is less than unity, and the second medium is said to be **optically rarer**, or less dense, than the first.

When light is incident in a vacuum (or in air) on a transparent surface, it is nearly always deflected toward the normal, so that the refractive indices of nearly all transparent media are greater than unity. Light can penetrate to a small depth into a metal, and in this case it is sometimes bent away from the normal (in the case of sodium, gold, and

silver) and sometimes toward the normal (in the case of platinum and iron). Thus, the refractive indices of sodium, gold, and silver are less than unity, whilst those of platinum and iron are greater than unity.

**Inverse Square Law.**—Imagine light to be emitted uniformly in all directions from a small source approximating in dimensions to a point. Then, whatever light may be, it is clear that, unless it can accumulate in space, the quantity emitted per second by the source will require one second to pass through each of a number of imaginary spherical surfaces, with centres at the source. If at any point it falls normally on a white surface, the illumination of the surface will obviously be proportional to the amount of light falling on unit area during one second. Hence, we may measure the illumination, at a distance $r$ from the source, by the amount of light passing per second through unit area of a sphere of radius $r$ and with centre at the source.

Let L be the light emitted per second by the source. Then, this light passes per second through the surface of the sphere of radius $r$, *i.e.* through an area $4\pi r^2$. Hence, the light passing per second through *unit area* of the same sphere is equal to—

$$L/4\pi r^2.$$

Thus, **the illumination at a given distance from a small source of light is inversely proportional to the square of the distance.**

Let us now suppose that the source of light is not infinitely small. We may, in imagination, divide it into a large number of elements, each indefinitely small.

The resultant illumination at any point is equal to the sum of the illuminations there, due to the various elements. The illumination due to any element varies inversely as the square of the distance; and a point beyond a certain finite distance from the source will be practically equidistant from all of the elements. For points beyond this distance the resultant illumination varies inversely as the square of the distance, just as in the case of an infinitely small source. On the other hand, for points at distances from the source, which are small in comparison with the linear dimensions of the latter, the resultant illumination is practically independent of the distance.

**Comparing Sources of Light.**—Let a small source emit light uniformly in all directions, the rate of emission being equal to $L_1$. The amount of light passing per second through unit area,

placed at right angles to the direction of the rays, at a distance $d_1$ from the source, is equal to $L_1/4\pi d_1^2$.

If another small source emits light at the rate of $L_2$, the amount of light passing through unit area at a distance $d_2$ from this source, is equal to $L_2/4\pi d_2^2$.

If white screens are placed at right angles to the rays at distances $d_1$ and $d_2$ from the respective sources, then these screens will be equally illuminated, and will appear equally bright, if—

$$L_1/4\pi d_1^2 = L_2/4\pi d_2^2,$$

$$\therefore \frac{L_1}{L_2} = \frac{d_1^2}{d_2^2}. \quad \ldots\ldots\ldots \quad (1)$$

This equation holds for all cases in which the linear dimensions of the sources are small in comparison with $d_1$ and $d_2$. It gives us a method of comparing the luminous emissivities of two sources of light. The comparison of luminous emissivities is termed **Photometry**, and an arrangement for effecting photometric measurements is termed a **Photometer.** Some of the best known photometers will now be described.

**Bouguer's Photometer.**—A translucent screen, AB (Fig. 6), made from ground glass or tissue paper, is mounted in a vertical plane, and a blackened opaque screen, CD, which is also vertical, is placed at right angles to AB, so that its edge, C, divides the translucent screen into two equal portions. Two sources of light, $L_1$ and $L_2$, are placed so as to illuminate opposite halves of the screen AB, while either source throws a shadow of the opaque screen CD on the part of AB illuminated by the other source. The screen is viewed from E. By varying the distances of $L_1$ and $L_2$ from the screen, the two shadows may be made equally bright; the ratio of luminous emissivities of the two sources are then obtained from (1) above, where $d_1$ and $d_2$ are the distances of the sources from C. The objection to this form of photometer is, that the two illuminated areas are not actually in contact, but are separated by a black line, corresponding to the thickness of the screen CD.

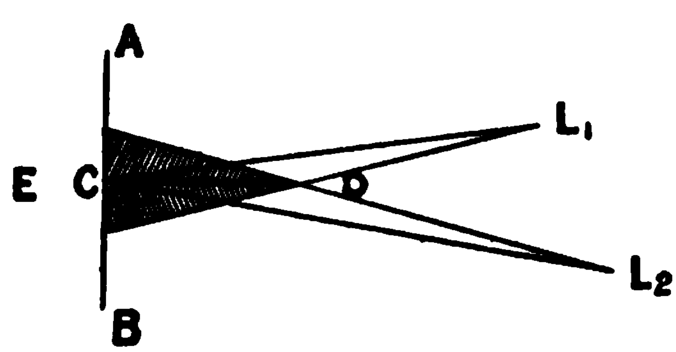

FIG. 6.—Bouguer's Photometer.

**Rumford's Photometer.**—A cylindrical rod, D (Fig. 7), such as a lead-pencil, is placed in front of a vertical screen, AB. The screen may be a sheet of unglazed white paper, when it should be viewed from the right; or a piece of tissue paper, or ground glass, when it should be viewed from the left. Two sources of light, $L_1$ and $L_2$, are placed to the right of AB in such positions that each throws a separate shadow of D on the screen, the area shaded from one source being illuminated by the other. The position of D should be adjusted so that the edges of the two shadows *just touch each other without overlapping*. By varying the distances of the two sources of light from the screen, the two shadows can be made equally bright, when equation (1) may be used. By the aid of Rumford's Photometer a very accurate comparison of two sources of light can be effected. It can be used in a room which is not quite dark.

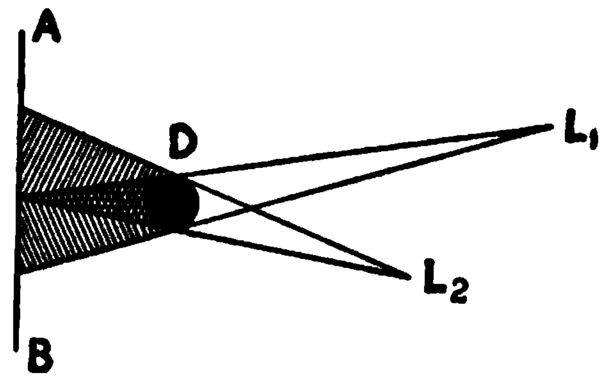

FIG. 7.—Rumford's Photometer.

**Bunsen's Grease-Spot Photometer.**—A screen is made from white unglazed paper, a small area of which has been greased to render it translucent. The grease spot should have sharp edges, and is preferably given the form of a star. The two sources of light are placed on opposite sides of the screen, and their distances are adjusted until the grease spot and the rest of the screen appear equally illuminated. Observations should be made on both sides of the screen. When the above adjustments have been made, equation (1) (p. 10) can be applied, where $d_1$ and $d_2$ are the respective distances of the light sources from the screen.

The theory underlying this experiment is very simple. Let $l_1$ and $l_2$ be the quantities of light per unit area falling on opposite sides of the screen from the sources $L_1$ and $L_2$ respectively. When unit quantity of light falls on the ungreased paper, let a fraction, $a$, be diffusively reflected, whilst the remainder $(1-a)$ is transmitted. Let $b$ and $(1-b)$ be the fractions of unit quantity of light diffusively reflected and transmitted per unit area of the grease spot. Then, if the screen is viewed from the side on which the source $L_1$ lies, the light reaching the eye from unit

area of the ungreased paper is proportional to $\{l_1a + l_2(1 - a)\}$, while that from unit area of the grease spot is proportional to $\{l_1b + l_2(1 - b)\}$. When the grease spot and the surrounding paper appear equally bright—

$$l_1a + l_2(1 - a) = l_1b + l_2(1 - b).$$

$$\therefore\ l_1(a - b) = l_2(1 - b - 1 + a) = l_2(a - b).$$

Since $a - b$ is not equal to zero, we may divide through by this quantity, when we obtain—

$$l_1 = l_2.$$

$$\therefore\ \frac{L_1}{d_1^2} = \frac{L_2}{d_2^2}.$$

**Joly's Photometer.**—A screen is formed from two similar plane parallel slabs of paraffin wax, placed face to face with a sheet of polished tinfoil interposed between them. The screen is placed between the sources of light, so that each slab is illuminated only by one source. The light is scattered as it traverses the wax, both before and after reflection from the tinfoil; consequently, when viewed sideways, the slabs appear bright, like the upper part of the wax of a lighted candle. The positions of the sources are adjusted till the two slabs appear equally bright.

**Light Standards.**—Since we have means of accurately comparing the luminous emissivities of different sources of light, it becomes important to select some standard source, the luminous emissivity of which may be taken as a unit. The conditions which such a standard should fulfil are as follows:—

1. Its luminous emissivity should be constant under the conditions usually attending photometric comparisons; or, if variation occurs, corrections should be applicable so as to reduce all observations to standard conditions.

2. The standard source should have the same luminous emissivity when set up independently by different observers, provided that certain specified conditions are fulfilled.

Few light standards fulfil these conditions, even approximately, while some make hardly any pretence to fulfilling them. Some of the most generally known standards will now be described.

**The British Standard Candle.**—This is a sperm candle, weighing six to the pound, and burning 120 grains per hour. The brightness of a candle-flame depends on the length and

shape of the wick (so that "snuffing" produces a considerable variation), the height of the flame, and even the temperature of the air and the amount of carbon-dioxide and water vapour present. The luminous emissivity of the British standard candle varies by about 20 per cent.

By the **candle-power** of a source we mean the ratio of the luminous emissivity of the source to that of the standard candle. Although candles have been universally abandoned as standards, the more trustworthy substitutes which have been adopted are generally defined as of so many candle-power.

**The Methven Standard.**—A screen with a rectangular aperture of definite dimensions is placed close to a coal-gas flame from an Argand burner (Fig. 8). The amount of light leaving the aperture varies considerably with the height of the flame, so that two wires (shown in the figure) are used to mark the standard height. The nature of the gas burnt appears to be of only secondary consequence. Though more trustworthy than a candle-flame, the Methven standard is liable to fluctuations amounting to 3 or 4 per cent.

**The Hefner-Alteneck Standard.**—This is the flame of a metal lamp in which amyl acetate is burnt. The height of the flame can be adjusted and measured. This standard is to be trusted to within about 2 per cent.

**The Vernon-Harcourt Pentane Standard.**—This is a ten candle-power standard, obtained by burning a mixture of air and vapour of pentane ($C_5H_{12}$). Liquid pentane is contained in a flat reservoir, A (Fig. 9), into which air is admitted by a stop-cock, S, while the heavy mixture of air and pentane vapour syphons over by way of the metal tube W, and the india-rubber tube V, at a rate regulated by the stop-cock T, and is finally burnt at a circular steatite burner with thirty holes, each between 1·25 and 1·5 mm. in diameter. The flame is drawn into a definite form, and its top is hidden from view, by a brass tube,

FIG. 8.—Methven Standard.

C, above the burner. Surrounding C is a wider tube, D, open at the bottom, but communicating at its upper end with the vertical tube E, which, in its turn, is connected with the central space of the steatite burner by way of the tube F. The air in the space between the tubes C and D becomes heated, and rises upwards. It becomes cooled in the connecting chamber between D and E, and descends, by way of the tube E, to the burner, thus supplying air to the middle of the latter at a constant rate. At the same time heat is supplied from the connecting chamber between D and E to the bracket supporting the reservoir A, so that the pentane is evaporated at a regular rate. The height of the flame is regulated, firstly by means of the stop-cocks S and T, and then more exactly by raising or lowering a cone N above the orifice of S. No glass chimney is used to surround the flame.

FIG. 9.—Vernon-Harcourt Pentane Standard.

The pentane is obtained by successively distilling light American petroleum (such as is known as gasoline) at the temperatures 55° C., 50° C., and 45° C. The distillate at 45° C. is shaken up with (1) strong sulphuric acid, and (2) with caustic soda solution. After this treatment it is again distilled, the portion which comes over between 25° C. and

40° C. being collected for use. It consists mostly of pentane, with small traces of higher and lower homologues, which do not affect the luminosity of the flame.

Pentane lamps, when constructed, according to specified conditions, by different persons, and set up by different experimenters, agree in candle-power to within about 0·1 per cent.

**Violle Standard.**—Violle recommends, as a standard of luminosity, one square centimetre of the surface of platinum heated to its melting point. This would doubtless be a most trustworthy standard if the difficulties attending its use could be overcome.

**The Electric Glow Lamp.**—An electric glow lamp, when the current and the voltage are maintained constant, gives a fairly constant light over a limited period of time. It has, however, been found impossible to construct a lamp so that its candle-power for a given current and voltage shall be known beforehand. Further, the filament of a glow lamp changes with use, and the inside of the bulb becomes blackened by carbon projected from the filament. The candle-power of a glow lamp also varies with the direction, with respect to the plane containing the filament, in which observations are made.

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**Plane and Solid Angles.**—Let AB (Fig. 10) be any straight line in the plane of the paper, and let P be any point in the same plane. Join AP, BP. Then the line AB is said to subtend the (plane) angle APB, at the point P. The angle APB may be measured in degrees, minutes, and seconds; but in theoretical investigations another method of measurement is preferable. If, with P as centre, we draw a number of circular arcs of different radii, extending between the lines AP and BP, then all of these arcs subtend the same angle at P, and the length of an arc, when divided by its radius, gives a ratio which is the same, whatever arc is chosen. Thus, the angle APB may be measured by the ratio—

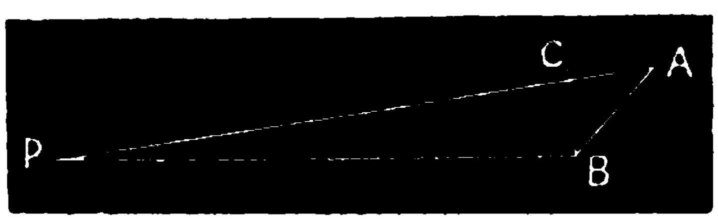

FIG. 10.—Plane Angle subtended by a Line.

$$\text{arc/radius} = \theta \text{ (say).}$$

$\theta$ is termed the **circular measure** of the angle APB. The unit of circular measure is termed a **radian**; it is the measure of an angle subtended by an arc equal in length to the radius. A radian is equal to 57°·3 (nearly). The circular measure of a right angle is equal to $\pi/2$ radians, where $\pi = 3{\cdot}14159$.

With P as centre, and PB as radius, describe the circular arc BC, cutting AP and BP in C and B. Then the circular measure of the angle APB is equal to—

$$\text{arc BC/radius PB}.$$

When the length BC is small in comparison with the length PB, the arc BC approximates to a straight line, and the figure ACB approximates to a triangle, with a right angle at C. Let AB = $c$, whilst BC = $a$, and PB = $r$; then—

$$a/c = \cos \text{ABC}, \text{ and } a = c \cos \text{ABC}.$$

Thus, circular measure of APB = $c \cos \text{ABC}/r$.

Let AB (Fig. 11) represent an area, supposed to be perpendicular to the plane of the paper; whilst P is a point in the plane of the paper. From numerous points on the boundary of AB draw straight lines to P. These lines, if numerous enough, generate a cone, with vertex at P. If

FIG. 11 – Solid Angle subtended by a Surface.

a number of spheres are described with P as centre, then from each of these an area will be cut off by the cone APB. These areas are proportional to the squares of the radii of the spheres to which they correspond, so that if any one of these areas is divided by the square of the radius of the corresponding sphere, the ratio so obtained will have the same value, no matter which area is chosen. From analogy with the method of measuring a plane angle, this ratio is said to measure the **solid angle** subtended at P by the surface AB.

With P as centre, and PB as radius, describe a sphere cutting the cone APB in the closed curve CB. The solid angle APB is then equal to—

$$\text{Area of CB}/(\text{distance PB})^2 = \omega \text{ (say)}.$$

When the linear dimensions of CB are small in comparison with PB, the surface CB will be approximately plane. Let $\phi$ be the angle of inclination between the surfaces AB and CB; this is conveniently measured by the angle between the normals to AB and CB. If the area of AB is equal to $a$, that of CB will be equal to $a \cos \phi$. Then, if PB = $r$, the solid angle, $\omega$, which AB subtends at P, is given by—

$$\omega = a \cos \phi / r^2.$$

Since the area of a hemisphere, of radius $r$, is equal to $2\pi r^2$, it follows that a hemisphere subtends a solid angle equal to $2\pi r^2/r^2 = 2\pi$, at the centre.

---

**Oblique Illumination.**—Let P (Fig. 11) be a point source of light, while CB is a small area of a spherical surface described with PB $= d$ as radius. Let AB be a small plane element of area inclined to CB at an angle $\phi$, both CB and AB being sections of the cone APB. Then it is obvious that the light which passes normally through CB will fall, approximately at an angle of incidence equal to $\phi$, on the surface AB. Let $l$ be the amount of light passing normally through unit area of CB. Then, if the area of AB is equal to $a$, the area of CB is equal to $a \cos \phi$, and the amount of light which passes normally through CB, and afterwards falls at an angle of incidence equal to $\phi$ on AB, is equal to $la \cos \phi$. The amount of light falling on unit area of AB is thus equal to—

$$la \cos \phi/a \doteq l \cos \phi.$$

If the source at P emits light at a rate L, $l = \mathrm{L}/4\pi d^2$. Then the illumination per unit area of AB is equal to—

$$\mathrm{L} \cos \phi/4\pi d^2.$$

From the above it will readily be understood that **the rates at which light falls on small surface elements, which subtend equal solid angles at the source, are equal.**

**Intrinsic Luminosity.**—Let us now consider the emission of light from an extended source, such as a sheet of platinum heated to incandescence by the electric current. Let us suppose that a very small element of the source, of area $a$, emits light normally at such a rate that *if the emission were uniform in all directions*, the rate of emission would be equal to L. The quantity of light derived from this element, which falls on a unit of area placed at right angles to the direction of the emitted rays, at a distance $d$ from the element, would be equal to—

$$\mathrm{L}/4\pi d^2,$$

and, since this amount of light is derived from an area $a$ of the source, that due to unit area of the source would be equal to—

$$\mathrm{L}/4\pi d^2 a.$$

The quantity $\mathrm{L}/4\pi a$ is termed the **intrinsic luminosity** of the source, and will be denoted by **L**. Since $4\pi$ is the solid angle

which a spherical surface subtends at the centre, we see that **the intrinsic luminosity of a source is the rate at which light is emitted per unit solid angle per unit area of the source.** We may also define the **unit quantity of light as that emitted per unit solid angle by a point source of one candle-power.** Then it is easily seen that at a perpendicular distance $d$ from a source of area A, and intrinsic luminosity L, the resultant illumination per unit area is equal to—

$$LA/d^2.$$

Now $A/d^2$ is the solid angle, $\omega$, which the surface of area A subtends at the point at a perpendicular distance $d$ from the surface. Thus, we see that **the illumination per unit area at a given point is equal to the intrinsic luminosity of the source, multiplied by the solid angle which the latter subtends at the point in question.**

The optical system of the eye will be considered in a later chapter. For the present it may be remarked that, when a luminous object is placed in front of the eye, light from the object, after passing through the pupil, forms an image on the interior back surface of the eye. The conditions which determine the magnitude of this image are essentially similar to those described in connection with the pin-hole camera (p. 5); the system of lenses with which the eye is provided serves merely to prevent the blurring of the image which would result in a pin-hole camera with an aperture as large as the pupil and a chamber as small as the eye. Thus, a given object will give rise to an ocular image of which the *linear dimensions* vary inversely as the distance of the object from the pupil (p. 5); the *area* of the image varies inversely as the square of the distance of the object from the pupil.

Let A′ be the area of the pupil; then the quantity of light which enters the eye from an object, of intrinsic luminosity L and area A, placed at a distance $d$ from the pupil, will be equal to—

$$LAA'/d^2.$$

Thus, the quantity of light which enters the eye from a given object is inversely proportional to the square of the distance of the latter from the pupil; and, as proved above, the area of the ocular image formed is also inversely proportional to the square of the distance of the object from the pupil. Consequently, the luminosity of the ocular image (which is proportional to the illumination per unit area) is constant, whatever may be the position of the object. In other words, **an object appears equally bright at all distances from the eye.**

The dimness of distant objects when seen through the atmosphere is due to the partial opacity of the latter. Further, the above reasoning implies the formation of a definite ocular image of the object; thus, the conclusions reached do not apply when the object is so distant that a definite ocular image cannot be formed (*e.g.* in the case of a star, or a very distant candle-flame).

Up to the present we have assumed that the object is viewed by means of rays sent off normally, the intrinsic luminosity of the surface for normal emission being equal to L. Now let $L_\phi$ be the intrinsic luminosity of the surface in a direction making an angle $\phi$ with the normal.

Then, if the surface has an area A, and is inclined at an angle $\phi$ to the line of vision, its distance from the eye being $d$, the quantity of light which enters the pupil from it is equal to—

$$L_\phi AA'/d^2.$$

Experience shows that **a luminous surface appears equally bright whatever may be its inclination to the line of vision.**

A surface of area A, inclined at an angle $\phi$, produces an ocular image of the same size as another of area $A \cos \phi$, perpendicular to the line of vision, the distances of the two surfaces from the eye being equal. Hence, for the ocular images to be equally bright—

$$L_\phi AA'/d^2 = LA \cos \phi \,.\, A'/d^2.$$
$$\therefore L_\phi = L \cos \phi.$$

Thus, **the intrinsic luminosity of a surface, for different directions of emission, varies as the cosine of the angle which the emitted light makes with the normal.**

**Visual Estimate of Luminosity.**—By placing a standard source of light at a suitable distance from a white screen, the intrinsic luminosity of the screen, which is determined by the quantity of light falling on unit area, can be adjusted at pleasure. It does not follow that a visual estimate of the luminosity will agree with the intrinsic luminosity of the screen. The intrinsic luminosity depends merely on physical conditions: a visual estimate depends, in addition, on the sensitiveness of the eye to light of various intensities.

Let the intrinsic luminosity of a white screen be L, and let $dL$ be the smallest increase in L which produces a difference distinguishable by the eye. Then we may say that increasing L by $dL$ produces unit difference in the visual luminosity of the

surface. From the result of experiments Weber proposed the law that $d\mathrm{L}/\mathrm{L}$ is constant, or that the smallest increase in the intrinsic luminosity of a surface which can be distinguished by the eye, is proportional to the original intrinsic luminosity of the surface. This law has been confirmed by Schirmer, for illuminations varying from that of 1 to 1000 candles at 1 cm. distance from the screen.

Weber's law may be investigated as follows. Let a white screen be illuminated by a standard source of light at a known distance from the screen; then the quantity of light falling on unit area of the screen becomes known; this quantity is proportional to L. Now let another standard source of light be placed at a considerable distance from the screen, adjusted so that it throws a shadow of a rod which is *just visible* on the screen. The quantity of light per unit area of the screen, derived from this second source, can be calculated; this is proportional to $d\mathrm{L}$. Hence, the ratio $d\mathrm{L}/\mathrm{L}$ can be determined for various values of L. Schirmer found that, at first, the value of $d\mathrm{L}/\mathrm{L}$ was equal to 1/128, but with practice he obtained the smaller value 1/217.

## Questions on Chapter I

1. How would you determine experimentally the quantity of light reflected at different angles by a piece of plane glass?

2. The sun's rays fall upon a small square mirror placed horizontally, and are received after reflection on a vertical screen. What will be the shape of the illuminated patch on the screen, and how will it vary when the distance between the screen and the mirror is altered?

3. A surface is being illuminated by a bright but distant lamp. How would you measure the intensity of the illumination of that surface?

## Practical

1. Find the proportion of the light emitted by a gas flame which is reflected at 45° from a plate of glass.

2. A flame and a Bunsen photometer disc are placed a given distance apart. Determine the reduction of the illumination at the disc when a piece of opaque glass of given size is placed between it and the flame at four given distances from the flame.

## CHAPTER II

### APPLICATIONS OF THE LAWS OF REFLECTION

**Introductory.**—The laws of reflection from a smooth surface have already been stated. Simple experiments illustrative of these laws may be found in most elementary works on light; but in order to establish them experimentally, somewhat complicated optical arrangements are necessary. On the other hand, the trustworthiness of the laws of reflection may be established, indirectly, by showing that the results deduced from them are in complete conformity with our experimental knowledge.

A smooth surface from which light is reflected is termed a **mirror**. A polished metallic surface reflects a large proportion of the light which falls on it, while the surface of a transparent medium reflects very little light; but in both cases the surface is termed a mirror. As a general rule, the reflecting surface, in order to be termed a mirror, should possess some simple geometrical form, such as a plane, or a part of a sphere, ellipsoid, or paraboloid.

### PLANE MIRRORS

**Formation of Image.**—Let A (Fig. 12) be a luminous point in the plane of the paper; and let MM′ be the section of a plane reflecting surface, supposed perpendicular to the plane of the paper, so that the normal BC to the surface lies in the plane of the paper. Let any two rays, AB, AB′, in the plane of the paper, respectively give rise to the reflected rays BD and B′D′; the rays BD and B′D′ will lie in the plane of the paper. Produce DB, D′B′, till they intersect at A′ (say). Then, since AB and BD are equally inclined to BC, they are equally

inclined to MM′. Therefore $\angle$ ABM = $\angle$ DBM′ = $\angle$ MBA′. Consequently, $\angle$ ABB′ = $\angle$ A′BB′. Similarly, $\angle$ AB′B = $\angle$ A′B′B. Further, the two triangles ABB′, A′BB′, have the common base BB′; consequently, these triangles are equal in all respects, and the side AB of one is equal to the side A′B of the other.

Join AA′. Then in the two triangles ABM, A′BM, the sides AB, A′B, are equal, the base MB is common, and the angles ABM and A′BM are equal. Hence, these triangles are equal in all respects, and AM = A′M. Further, since the angles BMA and BMA′ are equal, the line AMA′ is perpendicular to MM′.

FIG. 12.—Incident and Reflected Rays, Plane Mirror.

Thus, any two rays, AB, AB′, from a luminous point A, give rise to reflected rays, BD, B′D′, which follow the same paths, after leaving the surface MM′, as if they diverged from a point A′ behind the surface. **The point A′ lies on the line AM (produced), drawn from A perpendicular to the surface; and is as far behind the surface as the luminous point A is in front of it.** Since the specification of the point A′ does not involve the positions of the points B, B′, at which the rays are reflected, it follows that *all rays from* A, after being reflected from the surface MM′, follow the same paths as if they diverged from A′. Thus, to an eye placed above the mirror MM′, the reflected rays appear to diverge from the point A′. The point A′ is termed the **image** of A; since the rays reflected from the surface do not, in reality, pass through A′, but only appear to diverge from that point, A′ is termed the **virtual image** of A.

Any luminous object, when placed in front of a plane mirror, gives rise to a reflected image, the position of which can be easily determined. Let AB (Fig. 13) be the object, placed in front

FIG. 13.—Reflected Image, Plane Mirror.

of the mirror MM′. Then the point A of the object gives rise to an image A′, as far behind the mirror as A is in front of it. The position of B′, the image of B, is determined in a similar manner. Thus, it becomes evident that the reflected image is of the same size as the object. The actual paths of the rays reaching the eye from different points of the object will be readily understood from Fig. 13.

EXPT. 1.—Support a thin sheet of plate glass perpendicular to the bench, and place a pin upright in front of it. A reflected image of the pin is seen in the glass. Place a second pin, similar to the first, behind the sheet of glass, in such a position that it and the reflected image of the first pin appear to occupy the same position. When this adjustment has been accurately made, moving the eye from side to side produces no relative motion between the one pin and the image of the other; this condition is described by saying that there is no **parallax** between one pin and the image of the other. When the adjustment is imperfect, a motion of the eye from side to side causes the pin and the image to separate from each other; whichever of the two is farther from the eye will be displaced from the other in the same direction as that in which the eye moves. The reason of this can be easily seen by placing two fingers, one behind the other, in front of the eye, and then moving the latter from side to side. Having adjusted the second pin, measure its distance, and the distance of the first pin, from the glass. Also notice that, if the two pins are similar, one pin and the image of the other appear of the same size.

**Multiple Reflections.**—A pencil of light may undergo reflection at two or more plane mirrors before reaching the eye. Let us consider the case where it is successively reflected from two mirrors. After reflection at the first mirror, the light apparently proceeds from the image formed in that mirror; if this light falls on a second mirror, it will subsequently appear to proceed from a point which is the image of the first reflected image. A third reflection may then occur at the first mirror, in which case an image of the second reflected image is formed. Further reflections may take place, until an image is formed in such a position that light from it cannot be further reflected.

1. *Two Mirrors perpendicular to each other.*—Let O (Fig. 14) be a luminous point, situated between two mirrors, CM and CM′, which are perpendicular to each other. By direct reflection, an image $I_1$ is formed in the mirror CM. Part of the light apparently proceeding

from $I_1$ may reach the eye without further reflection, so that the virtual image $I_1$ may be seen. But some of the light from $I_1$ will be reflected from the mirror CM′, as shown in the figure; this light, on reaching the eye, appears to proceed from a point $I_2$, which is as far behind the plane of the mirror CM′ as $I_1$ is in front of it. To find the position of $I_2$, draw $OMI_1$ perpendicular to CM, and measure off $MI_1$ equal to MO; then draw $I_1I_2$ perpendicular to M′C produced, and mark the point $I_2$ in such a position that M′C produced bisects the line $I_1I_2$.

FIG. 14.—Images formed in two mutually Perpendicular Mirrors.

The image $I_2$ is below the mirror CM; and, consequently, light from it cannot suffer a further reflection.

By direct reflection in the mirror CM′, the object O gives rise to an image $I_1'$, and it is easily seen that the light from $I_1'$ which falls on CM gives rise to an image $I_2$, coinciding with that previously determined. Thus, three images, and no more, are formed by reflection in two mutually perpendicular mirrors.

2. *Two Mirrors inclined at any angle.*—Let $CM_1$, $CM_2$ (Fig. 15), represent two mirrors inclined to each other at an angle $M_1CM_2$, which, for simplicity, we shall suppose to be equal to $2\pi/n$, where $n$ is an integer. Let O be a luminous point lying between the mirrors. With C as centre, describe a circle passing through O. Then, if we mark a point $I_1$ on this circle, such that $I_1$ and O are on opposite sides of $CM_1$, and $M_1I_1 = M_1O$, it is easily seen that $I_1$ is the image of O formed by a single reflection in $CM_1$. For, from the geometry of the circle, a straight line joining O to $I_1$ will be perpendicular to $CM_1$, and $I_1$ is as far behind $CM_1$ as O is in front of it. Similarly, if we mark a point $I_2$ on the circle, such that $I_2$ and O are on opposite sides of $CM_2$, while $M_2I_2 = M_2O$, then $I_2$ is the image of O formed by a single reflection in $CM_2$.

Since $I_1$ is in front of the mirror $CM_2$, an image of it will be formed by reflection in $CM_2$. This image occupies the position marked $I_{12}$,

where $M_2I_{12} = M_2I_1$. From C draw the straight line CD, making the angle $DCM_2$ equal to $M_2CM_1$. Then CD is the image of the mirror $CM_1$, formed by a single reflection in $CM_2$; and it is easily seen that the images $I_2$ and $I_{12}$ occupy positions with respect to D, similar to those of O and $I_1$ with respect to $M_1$.

The reasoning used above may be extended to determine the images formed by three and more reflections. Draw the straight lines CE, CF, CG, CH, CK, and CL, dividing the circle into sectors each containing an angle equal to $M_1CM_2$, or $2\pi/n$; also produce $M_1C$ to $m_1$, and $M_2C$ to $m_2$. The images of $I_2$ and $I_{12}$, formed by reflection in $CM_1$, will respectively occupy the positions $I_{21}$ and $I_{121}$, equidistant from the point K. The images $I_{21}$ and $I_{121}$ give rise to the images $I_{212}$ and $I_{1212}$, by reflection in $CM_2$; and the images $I_{212}$ and $I_{1212}$ give rise to the images $I_{2121}$ and $I_{12121}$, by reflection in $CM_1$. The latter images are formed behind the plane of the mirror $CM_2$, and consequently no further reflection can occur. In general, when an image is formed within the space $m_1Cm_2$, no further reflection can occur. We can

FIG. 15.—Images formed in Inclined Mirrors.

now determine the total number of images formed. The lines $OM_1$, $OM_2$, OD, OE, . . . OK, and OL divide the circle $M_1DHL$ into $n$ sectors, each possessing an angle equal to $2\pi/n$. The sector $M_1CM_2$ contains the object; when $n$ is an odd integer (as in Fig. 15), each of the remaining $(n - 1)$ sectors contains one, and only one, image, with

the exception of the sector FCG, which contains two images. Hence, in general, when $n$ is odd, there will be $n$ images; but when the object O is midway between $M_1$ and $M_2$, it is easily seen that the images $I_{1212}$ and $I_{1212}$ will coincide, being situated at $m_1$; in that case there are only $(n - 1)$ separate images. If $n$ is an even integer, there will only be one image in each of the $(n - 1)$ sectors left after excluding $M_1CM_2$; in this case there are only $(n - 1)$ images formed.


FIG. 16.—Images formed in two Parallel Mirrors.

3. *Two Parallel Mirrors.*—Let $M_1M_1$, $M_2M_2$ (Fig. 16) be two plane mirrors, perpendicular to the line AB, and let O be an object placed between them. Draw lines perpendicular to AB, at distances apart equal to that between $M_1M_1$ and $M_2M_2$. An image $I_1$ is formed by reflection in $M_1M_1$, and O and $I_1$ respectively give rise to the images $I_2$ and $I_{12}$, by reflection in $M_2M_2$. The images $I_2$ and $I_{12}$ respectively give rise to the images $I_{21}$ and $I_{121}$, by reflection in $M_1M_1$, and so on. There would be an infinite number of images but for the circumstance that the light is weakened at each successive reflection, so that the images formed by a great number of reflections are too dim to be seen.

**Measurement of Small Deflections.**—In many physical experiments it is necessary to measure the deflection of a suspended system capable of rotation about a vertical axis. A pointer may be attached to the suspended system, and allowed to rotate above a circular scale; but in this case the length of the pointer must be considerable if very small angular deflections are to be measured, and the

FIG. 17.—Method of measuring Small Angular Deflection.

inertia of the pointer renders the time of oscillation of the suspended system inconveniently long. Poggendorff introduced an arrangement in which a small plane mirror is attached to the suspended system, and the image of a horizontal scale, reflected in this mirror, is viewed by the aid of a telescope. Let MM (Fig. 17) be the position of the mirror when the system is undeflected. The graduation A of the scale CD will form an image at E, and this is viewed by means of the telescope T, provided with cross-wires in the eye-piece. Let the suspended system rotate through a small angle, $\theta$, so that the mirror acquires the position $M_1M_1$; then the normal ON to the mirror rotates through an angle NOA $= \theta$. Draw OC, making an angle CON $= \theta$ with the normal. Then a light-ray CO from the point C will be reflected along the path OA, and the image of the graduation C of the scale will now be formed near E. To find the exact position of the image, draw a line from C perpendicular to $M_1M_1$ produced, and measure off a distance PQ = CP. The image of C will be formed at Q, and will be seen on the cross-wires of the telescope.

Since $\angle$ COA $= 2\theta$, we have—

$$AC/OA = \tan 2\theta.$$

When $\theta$ is small, $\tan 2\theta$ is equal to the circular measure of $2\theta$, to a very close approximation. In this case—

$$\theta = AC/2OA.$$

## Spherical Mirrors

**Definitions.**—A polished surface having the form of a portion of a sphere is termed a **spherical mirror**. The **centre of curvature** of the mirror is the centre of the sphere of which the reflecting surface forms a part. A spherical mirror is either concave or convex, according as the polished surface faces toward, or away from, the centre of curvature. The boundary of a mirror is generally circular. The middle point of the reflecting surface is termed the **pole** of the mirror. The diameter of the circular boundary of a mirror is termed the **aperture**. A section of a mirror, by a plane passing through the centre of curvature and the pole, is termed a **principal section**; and a straight line passing through the centre of curvature and the pole is termed

the **principal axis**, or, for brevity, the **axis** of the mirror. Strictly speaking, any straight line which passes through the centre of curvature, and intersects the surface of the mirror, might be termed an axis.

**Geometrical Conventions.**—The complete specification of a distance involves three elements: a numerical magnitude, measured in terms of some standard unit of length, such as the foot, inch, or centimetre; a direction; and lastly the point from which the distance is measured. If we prefix a positive sign (+) to distances measured in any given direction, then a negative sign (−) must be prefixed to distances measured in the opposite direction.

It is necessary to have a simple means of defining the position of a point on the axis of a mirror. **Distances are measured along the axis from the pole of the mirror; when the direction of measurement is opposite to that in which the incident light travels, the distance is positive; distances measured in the reverse direction are negative.**

**The radius of curvature** of a mirror is the radius of the sphere of which the reflecting surface forms a part. It is equal to the distance from the pole to the centre of curvature. Since the reflecting surface, when used as a mirror, must be turned toward the source of light, it follows that **the radius of curvature of a concave mirror is positive, whilst that of a convex mirror is negative.**

---

**Trigonometrical Ratios.**—Let AB, AC (Fig. 18) be any two lines, inclined at an angle BAC = $\theta$. In theoretical investigations, $\theta$ must be understood to signify the circular measure of the angle BAC (p. 15). From any point, B, on one of the lines, draw BC perpendicular to the other line, thus forming a right-angled triangle. The side AC, joining the point A to the right angle C, is termed the **base**, while the side CB, perpendicular to the base, is termed the **perpendicular**, and the remaining side AB is termed the **hypotenuse**. Then, for a given value of $\theta$, the ratios—

FIG. 18.—Trigonometrical Ratios.

$$\frac{\text{perpendicular}}{\text{hypotenuse}}, \quad \frac{\text{base}}{\text{hypotenuse}}, \quad \frac{\text{perpendicular}}{\text{base}},$$

are constant. Definite names are given to these ratios, as follows:—

$$\frac{\text{perpendicular}}{\text{hypotenuse}} = \text{sine of angle BAC} = \sin\theta.$$

$$\frac{\text{base}}{\text{hypotenuse}} = \text{cosine of angle BAC} = \cos\theta.$$

$$\frac{\text{perpendicular}}{\text{base}} = \text{tangent of angle BAC} = \tan\theta.$$

The case where $\theta$ is very small demands particular attention. Let AB, AC (Fig. 19) be two straight lines inclined at a small angle. Draw BC perpendicular to AC, and with A as centre, and AB as radius, describe the circular arc BD, cutting the lines in B and D. Then the circular measure of the angle BAC is equal to—


FIG. 19.—Trigonometical Ratios of very small Angle.

$$\text{arc BD/AB}.$$

The perpendicular BC is nearly equal in length to the arc BD, and the two become more nearly equal as the angle BAC is diminished. Thus, for small values of $\theta$—

$$\sin\theta = \text{BC/AB} = \text{BD/AB} = \theta,$$

to a close approximation. Similarly—

$$\tan\theta = \text{BC/AC} = \text{BD/AB} = \theta \text{ (nearly)};$$

and

$$\cos\theta = \text{AC/AB} = \text{AC/AD} = 1 \text{ (nearly)}.$$

The following table gives the circular measures, sines, cosines, and tangents of some small angles:—

| Angle (in degrees). | Circular measure ($\theta$). | sin $\theta$. | tan $\theta$. | cos $\theta$. |
|---|---|---|---|---|
| 1° | 0·0175 | 0·0175 | 0·0175 | 0·9998 |
| 2° | 0·0349 | 0·0349 | 0·0349 | 0·9994 |
| 5° | 0·0873 | 0·0872 | 0·0875 | 0·9962 |
| 10° | 0·1745 | 0·1736 | 0·1763 | 0·9848 |
| 15° | 0·2618 | 0·2588 | 0·2679 | 0·9659 |
| 20° | 0·3491 | 0·3420 | 0·3640 | 0·9397 |

Thus, **for a small angle, the sine and tangent are each equal to the circular measure, to a close approximation; while the cosine approximates to unity.** We shall often find it convenient to use the sine or tangent of a small angle as a first approximation to the circular measure of the angle.

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**Reflection from Concave Mirror.**—Let APB (Fig. 20) be a principal section of a concave mirror, of which P is the pole, and C is the centre of curvature. Let the radius of curvature, PC, be equal to $r$ (a positive quantity), and let O be a luminous point on the axis. Let OA be a ray in the plane of the paper, incident on the mirror at A; it is required to determine the corresponding reflected ray. Join AC. Then, since AC is a radius to the spherical surface of which APB is a section, the small element of area surrounding A will be perpendicular to AC, or AC is the normal to the surface at A. Then, since the incident ray OA and the normal AC lie in the plane of the paper, the reflected ray AI also lies in the plane of the paper, and $\angle$ IAC = $\angle$ OAC = $i$ (say) (p. 6).

FIG. 20.—Incident and Reflected Rays, Concave Mirror.

We must now determine the position of the point I where the reflected ray AI cuts the axis. Let the points O and I be at distances equal to $u$ and $v$ respectively from the pole P; in the figure, $u$ and $v$ are both positive. From A draw AD perpendicular to the axis, meeting the latter in D. When AD is small in comparison with the radius of curvature of the surface, the distance PD will be small in comparison with $u$, $v$, or $r$, and we may, without committing any appreciable error, write—

$$DC = r, \ DO = u, \ DI = v.$$

Let $\angle AOC = O$, $\angle ACI = C$, and $\angle AIP = I$. Then, since I is the external angle of the triangle AOI, of which the angle IAO is equal to $2i$—

$$I = 2i + O. \quad \ldots \ldots \ldots \quad (1)$$

Similarly, from the triangle AOC—

$$C = i + O. \quad \ldots \ldots \ldots \quad (2)$$

From (2)—

$$2C = 2i + 2O. \quad \dots\dots\dots \quad (3)$$

Subtracting (1) from (3), we obtain—

$$2C - I = O\,; \therefore I + O = 2C. \quad \dots\dots \quad (4)$$

So far the reasoning used has been perfectly rigid. We must now substitute approximate values of the angles O, I, and C in (4). When these angles are small, we may measure them by their tangents (p. 30). Thus, if DA $= y$—

$$O = DA/DO = y/u,$$

while I $= y/v$, and C $= y/r$.

Then, from (4)—

$$\frac{y}{v} + \frac{y}{u} = \frac{2y}{r}.$$

Dividing through by $y$, we find that—

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}. \quad \dots\dots\dots \quad (5)$$

Equation (5) does not involve $y$, so that the same value of $v$ is obtained at whatever point on the mirror the reflected ray originates. In other words, **all rays from O, after reflection at the mirror, converge toward a single point I on the axis, at a distance $v$ from the pole; and the value of $v$ is determined by** (5). After passing through the point I, the reflected rays diverge, and on reaching the eye, appear to originate at a luminous point at I. Thus I is the **reflected image** of O; and since the light actually passes through I, the image is said to be **real**. A small piece of white paper, held perpendicular to the axis at I, will exhibit a bright spot where the real image is situated.

It must be remembered that equation (5) is only true when the angles O, I, and C are so small that the tangents may be substituted for the circular measures of the angles; in other words, when the ratio of the semi-aperture to the radius of curvature of the mirror is small.

**Reflection from Convex Mirror.**—Let APB (Fig. 21) be a principal section of a convex mirror, of which P is the pole, and C is the centre of curvature. Let the radius of curvature, PC, be equal to $r$ (a negative quantity), and let O be a luminous point on the axis, at a distance PO $= u$ from the pole. Let a ray OA

in the plane of the paper meet the mirror at A. Draw the radius CA, and produce this to E; then the reflected ray AR lies in the plane of the paper, and $\angle$ EAR $= \angle$ EAO $= i$ (say).

Produce RA to cut the axis in the point I, at a distance $v$ (a negative quantity) from P; it is required to determine the position of the point I on the axis.

FIG. 21.—Incident and Reflected Rays, Convex Mirror.

Since $\angle$ RAO ($= 2i$) is the external angle of the triangle AIO, we have—

$$2i = \text{I} + \text{O} \quad . \quad . \quad . \quad (6)$$

Similarly, from the triangle ACO—

$$i = \text{C} + \text{O}; \ \therefore \ 2i = 2\text{C} + 2\text{O} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Subtracting (6) from (7), we obtain—

$$2\text{C} + \text{O} - \text{I} = 0; \ \therefore \ \text{I} - \text{O} = 2\text{C} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It must be remembered that the angles O, I, and C are all essentially positive. Draw AD perpendicular to the axis, and let DA $= y$. Then, when the angles O, I, and C are small—

$$\text{O} = \frac{y}{u}, \quad \text{I} = -\frac{y}{v}, \quad \text{and} \quad \text{C} = -\frac{y}{r},$$

negative signs being prefixed to terms involving the negative quantities $v$ and $r$, so as to obtain positive values for the angles I and C. Then, from (8)—

$$-\frac{y}{v} - \frac{y}{u} = -\frac{2y}{r}. \quad \therefore \ \frac{1}{v} + \frac{1}{u} = \frac{2}{r} \quad . \quad . \quad . \quad . \quad (9)$$

Since equation (9) does not involve $y$, it follows that **all rays from O, after reflection at the mirror, appear to diverge from a single point I on the axis.** Thus, the point I is the **reflected image** of O; since the light-rays do not actually pass through I, but only appear to diverge from that point, the image is said to be **virtual**.

Equation (9) is of exactly the same form as equation (5), so that, when due attention is paid to signs, a single equation applies to both concave and convex mirrors.

**Conjugate Foci.**—Rays from an object at O (Figs. 20 and 21), after reflection at the mirror APB, form an image at I. The object need not, of necessity, be an actual luminous point, but may be an image formed by reflection from another mirror. If the object is on the positive side of the mirror APB, $u$ is positive, and the object is real. If the object is an image formed by reflection at another mirror, it may be situated on the negative side of APB, in which case $u$ is negative, and the object is virtual. Further, since a ray, IA (Fig. 20) or RA (Fig. 21), will be reflected along the path AO, it follows that an object (real or virtual) at I will give rise to an image at O. Thus, the points O and I are such that an object at one of them gives rise to an image at the other. The points O and I are termed **conjugate foci** with respect to the mirror.

**Principal Focus.**—Let us now suppose that the object is at an infinite distance from the mirror. In this case the incident rays are parallel to the principal axis (p. 3), and since $u = \infty$, $1/u = 1/\infty = 0$. Let $f$ be the value of $v$ corresponding to $u = \infty$. Then, from (5) or (9)—

$$\frac{1}{f} = \frac{2}{r}. \quad \therefore f = \frac{r}{2}. \quad \ldots\ldots\ldots \quad (10)$$

Thus, **rays originally parallel to the axis, when reflected at the mirror, form a focus (real or virtual) at a distance $f$ from the pole.** The distance $f$ is termed the **focal length** of the mirror; its value is equal to half the radius of curvature. When the radius of curvature is positive (concave mirror), the focal length of the mirror is positive, and rays, originally parallel to the axis, are brought to a real focus after reflection at the mirror. When the radius of curvature is negative (convex mirror), the focal length is negative, and rays, originally parallel to the axis, diverge from a virtual focus after reflection at the mirror. The point on the axis, at a distance $f$ from the pole, is termed the **principal focus** of the mirror.

Since the principal focus is conjugate to a point at an infinite distance, it follows that **rays which diverge from the real principal focus ($f$ positive) or converge toward the virtual principal focus ($f$ negative) are rendered parallel to the axis after reflection at the mirror.**

Equations (5) and (9) may now be written—

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

**Relative Positions of Image and Object.**—Let us now examine the equation—

$$\frac{1}{v} = \frac{2}{r} - \frac{1}{u}, \quad \ldots \ldots \ldots \quad (11)$$

in order to determine the changes in the position of the image as the object is moved along the axis.

1. *Let $r$ be positive (concave mirror).*—When the object is at an infinite distance, the image is formed at the principal focus, at a distance $r/2$ from the pole.

As the object moves along the axis toward the mirror, the value of $u$ diminishes, and that of $1/u$ increases, with the result that the value of $1/v$ diminishes, and $v$ increases. When $u > r$, $v < r$.

Where $u = r$ (*i.e.*, when the object is at the centre of curvature of the mirror), $v = r$, or the image is formed at the same point as the object. In this case each ray from the object falls normally on the mirror, and after reflection, retraces its previous course.

When the value of $u$ lies between $r/2$ and $r$, the value of $v$ is greater than $r$, and lies between $r$ and $\infty$. When $u = r/2$, $v = \infty$.

When $u < r/2$ (*i.e.*, when the object is nearer to the mirror than the principal focus), $1/u > 2/r$, and $v$ is negative, so that the image is virtual, being formed on the negative side of the mirror.

When $u = 0$ (*i.e.*, when the object is formed at the pole of the mirror), $v = 0$. For, from (11)—

$$\frac{1}{v} = \frac{2u - r}{ur}. \quad \therefore v = \frac{ur}{2u - r}.$$

Substituting $u = 0$, we find that—

$$v = \frac{0}{0 - r} = 0.$$

When $u$ is slightly less than $\frac{r}{2}$, $v$ approximates to $-\infty$. Thus, as $u$ is varied from $r/2$ to $0$, $v$ varies from $-\infty$ to $0$, or the image moves up to the mirror from an infinite negative distance.

When $u$ is negative (*i.e.*, when the object is virtual), $v$ is positive, and its value changes from $0$ (when $u = 0$), to $r/2$ (when $u = -\infty$). Thus, as the virtual object moves away from the mirror in the negative direction to an infinite distance, the image moves in the positive direction along the axis, from the pole to the principal focus.

2. *Let $r$ be negative (convex mirror).*—When $u$ is positive (*i.e.*, when the object is real), both terms on the right of (11) are negative, and therefore $v$ is negative. Thus, **a real object always gives rise to**

**a virtual image in a convex mirror.** When $u = +\infty$, the image is formed at the principal focus. As $u$ is diminished, remaining positive, the numerical value of $1/v$ increases, and that of $v$ decreases. Thus the virtual image of a real object, formed by reflection at a convex mirror, never lies farther from the mirror than the principal focus. When $u = 0$, then $v = 0$. **The image moves from the principal focus to the pole, as the real object moves up from $+\infty$ to the pole.** When the object is virtual, $u$ is negative. When the virtual object is situated at the principal focus, all the reflected rays are parallel to the axis, and the image is formed at $+\infty$. Thus, as the virtual object moves from the pole ($u = 0$) to the principal focus ($u = f = r/2$), the image moves from the pole along the axis to $+\infty$, and the image is thus real (compare Fig. 21, supposing RA to be the incident ray). Since $u$ is numerically smaller than $r/2$, the sign of $1/v$ is determined by that of the second term to the right of (11), which is positive.

When $u$ is negative, and numerically greater than $f$, the sign of $1/v$ in (11) is determined by the sign of $2/r$, which is negative. In this case $v$ is negative, and the image is virtual. As the numerical value of $u$ exceeds that of $f$, the numerical value of $1/v$ increases, and therefore that of $v$ diminishes, so that the virtual object moves up from $-\infty$ toward the mirror. When $u = r$, the image and object coincide, the rays being reflected normally from the mirror. Thus, as $u$ increases (numerically) from $r/2$ to $r$, the image moves up from $-\infty$ to $r$.

When $u$ is numerically greater than $r$, $v$ is numerically less than $r$, but greater than $r/2$. Thus, as the virtual object moves from the centre of curvature to $-\infty$, the virtual image moves toward the lens from the centre of curvature to the principal focus.

The student should draw a diagram, showing the paths of the reflected rays, for each of the cases considered above.

**Object of Finite Dimensions.**—We have, up to the present, supposed the object to be a luminous point, or the image of a luminous point, situated on the axis of the mirror. We must now determine the nature of the image when the object is of finite (but small) dimensions, and lies near the axis, in a plane perpendicular to the latter.

Let OA (Fig. 22) be the object lying in the plane of the paper; one extremity, O, being on the axis OE of a mirror of which the pole is at E, while OA is perpendicular to OE. Let DEG be the section of a plane perpendicular to the axis OE, and passing through the pole E. Since the aperture of the mirror is supposed to be small in comparison with the radius of curvature,

the surface of the mirror will approximately coincide with the plane DEG, and if we arrange that the reflected rays pass through the proper points on the axis, we may draw them from the points in the plane DEG cut by the corresponding incident rays. The plane DEG may be termed the **Principal Plane** of the mirror. Let C be the centre of curvature, and F the principal focus, of the mirror, which is supposed to be concave, so that EF = EC/2. The image of the extremity O of the object will lie on the axis OE, as already proved; we shall now determine the

FIG. 22.—Graphical Construction for Image, Concave Mirror.

image of the extremity A of the object, after doing which the complete image may be drawn. To find the image of the point A, we must determine the point of intersection of any two reflected rays originally derived from A.

(1) The incident ray AD, parallel to the axis, gives rise, on reflection at the mirror, to the ray DFB, passing through the principal focus F (p. 33).

(2) The incident ray ACG, passing through the centre of curvature, is reflected back along its previous path (p. 34).

The point of intersection, B, of the reflected rays DB and GB, gives the image of the point A. A straight line, BI, drawn perpendicular to the axis, gives the complete image of OA.

It is sometimes convenient to use rays other than those employed above. The paths of two more rays are easily determined.

(3) The incident ray AFK, passing through the principal focus F, gives rise on reflection to a ray KB, parallel to the axis (p. 33).

(4) The ray AE, incident at the pole of the mirror, gives rise to a

reflected ray EB, such that ∠ BEO = ∠ AEO. This follows from the circumstance that the axis is normal to the mirror at E.

It will subsequently be shown that the reflected rays DB, EB, KB, and GB, all intersect in a single point, B. The intersection of *any two* of these rays gives the image of the point A.

Fig. 23 shows the construction when the mirror is convex.

(1) The ray AD, parallel to the axis, gives rise to a reflected ray which virtually proceeds from the principal focus F.

(2) The ray AG, directed toward the centre of curvature C, is reflected back along its previous path.

FIG. 23.—Graphical Construction for Image, Convex Mirror.

(3) The ray AR, directed toward the principal focus F, gives rise to a reflected ray parallel to the axis.

(4) The ray AE, reflected at the pole, gives rise to a ray EL such that ∠ AEO = ∠ LEO.

The various reflected rays, when produced backward, intersect in a point B on the negative side of the mirror. The point B is the image of A, and the complete image of OA is found by drawing a line BI perpendicular to the axis.

Fig. 22 shows that when the mirror is concave, and the real object lies beyond the principal focus, the image is real and inverted. When the object lies nearer to the mirror than the principal focus, the ray reflected parallel to the axis is that which, when produced backwards, passes through the principal focus F. Similarly, the ray which, when produced backwards, passes through the centre of curvature, is reflected back along its

previous path. The student should draw a diagram showing the formation of the image in this case. It will be found that the image is virtual and erect.

Fig. 23 shows that, when the mirror is convex, a real object gives rise to an image which is virtual and erect.

**Magnification.**—The ratio of the linear dimensions of image and object is termed the **magnification**. Thus, in Figs. 22 and 23, the magnification is equal to the ratio IB/OA. When the image is erect (Fig. 22), the distances IB and OA are measured from the axis in the same direction, and both have similar signs, so that the magnification is positive. When the image is inverted (Fig. 23), the distances IB and OA are measured from the axis in opposite directions, so that if one is positive, the other must be negative, and the magnification is negative. We must now find expressions for the magnification in terms of the quantities $u$, $v$, $f$, and $r$. For this purpose, Fig. 22 will be referred to, since there the quantities mentioned are all positive. The student should also verify each result with respect to Fig. 23.

In Fig. 22, let OA = $\mathbf{o}$, while IB = $\mathbf{i}$. In the figure, the ratio $\mathbf{i}/\mathbf{o}$ is negative. Let EF $= f$, EC $= r$, EI $= v$, and EO $= u$.

1. The right-angled triangles AEO and BEI are similar, since $\angle$AEO = $\angle$BEI. Thus, since $\angle$IEB = $-\angle$OEA—

$$IB/EI = -OA/EO.$$

$$\therefore \frac{\mathbf{i}}{\mathbf{o}} = -\frac{v}{u}, \quad \ldots\ldots\ldots \quad (12)$$

the negative sign being introduced for reasons explained above. Equation (12) determines the magnification in terms of $v$ and $u$.

2. The right-angled triangles AFO and KFE are similar, having equal angles at F. Thus—

$$EK/FE = OA/FO.$$
$$FO = EO - EF = u - f.$$
$$FE = -EF = -f.$$

EK = IB = $\mathbf{i}$, since KB is parallel to the axis.

Then, $\mathbf{i}/(-f) = \mathbf{o}/(u - f)$.

$$\therefore \frac{\mathbf{i}}{\mathbf{o}} = -\frac{f}{u-f} \quad \ldots\ldots\ldots \quad (13)$$

Equation (13) determines the magnification in terms of $u$ and $f$.

3. The right-angled triangles DEF and BIF are similar, having equal angles at F. Thus—

$$\begin{aligned}&\mathrm{IB/FI} = \mathrm{ED/FE}.\\&\mathrm{FI} = \mathrm{EI} - \mathrm{EF} = v - f.\\&\mathrm{ED} = \mathrm{OA} = \mathrm{o}, \text{ since DA is parallel to the axis.}\\&\mathrm{FE} = -\ \mathrm{EF} = -f.\end{aligned}$$

Then, $\mathrm{i}/(v - f) = \mathrm{o}/(-f)$.

$$\therefore\ \frac{\mathrm{i}}{\mathrm{o}} = -\frac{v - f}{f}. \qquad \ldots\ldots\ldots \qquad (14)$$

Equation (14) determines the magnification in terms of $v$ and $f$.

4. The right-angled triangles ACO and BCI are similar, having equal angles at C. Thus—

$$\begin{aligned}&\mathrm{IB/CI} = \mathrm{OA/CO}.\\&\mathrm{CI} = -\ \mathrm{IC} = -\ (\mathrm{EC} - \mathrm{EI}) = -\ (r - v).\\&\mathrm{CO} = \mathrm{EO} - \mathrm{EC} = u - r.\end{aligned}$$

Then, $-\ \mathrm{i}/(r - v) = \mathrm{o}\ (u - r)$.

$$\therefore\ \frac{\mathrm{i}}{\mathrm{o}} = -\frac{r - v}{u - r}. \qquad \ldots\ldots\ldots \qquad (15)$$

Equation (15) determines the magnification in terms of $u$, $v$, and $r$.

If the reflected rays DB, EB, KB, GB, all intersect in a single point, B, the expressions obtained for the magnification in (12), (13), (14), (15) must all be equivalent. It can be proved that this is the case. For instance, since—

$$1/v = 1/f - 1/u,$$

$$v = \frac{uf}{u - f}. \quad \therefore\ -\frac{v}{u} = -\frac{f}{u - f},$$

which proves the equality between (12) and (13).

Similarly—

$$u = \frac{vf}{v - f}. \quad \therefore\ -\frac{v}{u} = -\frac{v - f}{f},$$

which proves the equality between (12) and (14).

Since $1/v + 1/u = 2/r$,

$$1/v - 1/r = 1/r - 1/u,$$

and—

$$\frac{r - v}{vr} = \frac{u - r}{ur}. \quad \therefore\ -\frac{v}{u} = -\frac{r - v}{u - r},$$

which proves the equality between (12) and (15). The equality of (12),

(13), (14), (15) proves that the rays DB, EB, KB, GB, intersect in a single point B. We may now write—

$$\frac{\mathbf{i}}{\mathbf{o}} = -\frac{v}{u} = -\frac{f}{u-f} = -\frac{v-f}{f} = -\frac{r-v}{u-r}.$$

Equation (12) shows that when the object is real ($u$ positive) the ratio $\mathbf{i}/\mathbf{o}$ is negative, and the image is inverted, when $v$ is positive (real image); but $\mathbf{i}/\mathbf{o}$ is positive, and the image is erect, when $v$ is negative (virtual image). When the image is farther from the lens than the object, the magnification is greater than unity, or the image is larger than the object; otherwise the image is smaller than the object, except when $v = u$, when the image and object are equal in size.

Equation (13) shows that with a *concave mirror* ($f$ positive), when the object is beyond the principal focus ($u > f$), the image is inverted. When $u = f$, the denominator of the quantity to the right of (12) becomes equal to zero, and the magnification becomes infinitely great. The image is, in this case, of infinite size, but situated at an infinite distance from the mirror. When $u$ is positive, but less than $f$, the magnification becomes positive, and the (virtual) image is erect. When $u$ is negative, the magnification is always positive, and the image is erect.

Equation (13) shows that with a *convex mirror* ($f$ negative) the image is always erect when $u$ is positive. If $u$ is negative, the image is erect when $u$ is numerically less than $f$, but inverted when $u$ is numerically greater than $f$.

**Intrinsic Luminosities of Image and Object.**—Let an object consist of a small luminous surface of area A, placed perpendicular to the axis of a mirror, at a distance $u$ from it. Let the reflected image be formed at a distance $v$ from the mirror. Then, since the *linear dimensions* of the image and object are in the ratio $v/u$, their *areas* are in the ratio $v^2/u^2$. Thus, the area of the image is $v^2\mathrm{A}/u^2$.

When unit quantity of light falls on the mirror, let a quantity $k$ be regularly reflected, the rest being absorbed or lost in some other manner. The value of $k$ must always be less than unity.

If the mirror has an area A′, it subtends a solid angle $\mathrm{A}'/u^2$ at the object. Let L be the intrinsic luminosity (p. 18) of the object. Then the quantity of light falling on the mirror per second is equal to $\mathrm{LAA}'/u^2$, and the quantity reflected per second to form the image is equal to $k\mathrm{LAA}'/u^2$. The solid angle embraced by a pencil converging from the mirror to a

point of the image (if the latter is real), or diverging from a point of the image (if the latter is virtual), is equal to $A'/v^2$. In either case $A'/v^2$ is equal to the solid angle embraced by the complete pencil from a point of the image. Now, the intrinsic luminosity of the image is equal to the rate of emission of light per unit solid angle per unit area of the image (p. 18). Thus, the intrinsic luminosity of the image is equal to—

$$\frac{kLAA'}{u^2} \div \left(\frac{A'}{v_2} \times \frac{v^2A}{u^2}\right) = \frac{kLAA'}{u^2} \times \frac{u^2}{AA'} = kL.$$

Thus, since $k$ is always less than unity, **the intrinsic luminosity of the image is always less than that of the object.** Since the visual estimate of the brightness of an object or image depends on the intrinsic luminosity and not on the distance (p. 18), it follows that **an image formed by reflection from a mirror can never appear brighter than the object.**

This law only applies to cases where a definite image is formed (p. 19). Thus, the image of the moon in a reflecting telescope never appears brighter than the moon itself. On the other hand, the image of a star in a telescope becomes smaller as the aperture of the mirror is increased, quite irrespective of the focal length of the mirror[1]; the stars are so far away that a *true* image of one would approximate to a mathematical point. Hence, since the light from a star is concentrated over a smaller area in proportion as the aperture of a reflecting telescope is increased, the apparent brightness of the image of the star is also increased.

A luminous object radiates light in all directions, so that rays from it pass through all points of the pupil. The rays from an image are confined within a solid angle equal to $A'/v^2$. If this solid angle is so small that rays do not pass through all points of the pupil, the image will appear less bright than the object, even if $k = 1$.

## Ellipsoidal and Paraboloidal Mirrors

**Aberration.**—The results obtained (pp. 30 to 32) from the application of the laws of reflection to spherical mirrors are obviously only approximately true; they apply only when the incident and reflected rays are nearly parallel to the axis. If

[1] See p. 447.

the aperture of the mirror is small, the results obtained will apply to all the rays falling on the mirror; but when the aperture is large, the whole of the rays derived from a luminous point on the axis will not, after reflection, pass through a single point on the axis. This departure from the approximate laws already developed is termed **spherical aberration**; the consequences of this departure will be investigated in a succeeding chapter.

It is impossible to design a mirror of such a form that rays from *any* point on the axis shall, after reflection, form a point focus, real or virtual, on the axis. But it is possible to construct a mirror which shall bring rays, derived from a *particular* point on the axis, to a point focus on the axis. Such a mirror is said to be **aplanatic**, and the conjugate point foci are termed its **aplanatic foci**.

**Ellipsoidal Mirror.**—Let ABC (Fig. 24) be an arc of the ellipse ABCD, of which $F_1$ and $F_2$ are the geometrical foci. The diameter $BF_2F_1D$, passing through the foci, is termed the major axis of the ellipse. If we suppose Fig. 24 to rotate about the axis BD, the arc ABC will generate a surface constituting part of an ellipsoid of revolution. We may therefore consider the arc ABC to constitute a principal section of an ellipsoidal mirror.

FIG. 24.—Ellipsoidal Mirror.

It will now be shown that **any ray, $F_1E$, derived from one of the foci, will be reflected along $EF_2$, so as to pass through the remaining focus.** Thus $F_1$ and $F_2$ are the aplanatic foci of the mirror of which ABC is a principal section.

Let E be any point on the ellipse. Join $F_1E$ and $EF_2$. Then it must be shown that the lines $F_1E$ and $EF_2$ are equally inclined to the small element of the curve in the neighbourhood of E. In this case the lines $F_1E$ and $EF_2$ will be equally inclined to the normal at E, and if $F_1E$ is an incident ray, $F_2E$ will be the reflected ray.

Let G be a point close to E. Join $F_1G$ and $GF_2$. Then, if the arc EG is extremely small, it will approximate to a straight line, and the lines $F_1E$ and $F_1G$ will be approximately parallel, as will be the lines $F_2G$ and $F_2E$. With $F_1$ as centre, and radius $F_1E$, describe the arc EH. With $F_2$ as centre, and radius $F_2G$, describe the arc GK. When GE is extremely small, the figures GEH and EGK approximate to triangles with right angles at H and K respectively. Also, since $F_1E$ and $F_1G$ are approximately parallel, the angle EGH measures the inclination of $F_1G$ or $F_1E$ to the arc EG. Similarly, the angle GEK measures the inclination of $EF_2$ or $GF_2$ to the arc EG.

By a fundamental property of the ellipse, the sum of the distances of a point from the foci $F_1$ and $F_2$ is constant for all points on the curve. Thus—

$$F_1E + EF_2 = F_1G + GF_2, \text{ or } F_1E + EK + KF_2 = F_1H + HG + GF_2.$$

But $F_1E = F_1H$, and $KF_2 = GF_2$.

$$\therefore \ EK = HG.$$

Now—

$$\left.\begin{array}{l} \cos GEK = EK/EG \\ \cos EGH = HG/EG \end{array}\right\} \begin{array}{l} \therefore \ \cos GEK = \cos EGH, \\ \text{and } \angle GEK = \angle EGH. \end{array}$$

Thus, the lines $F_1E$ and $EF_2$ are equally inclined to the element of the curve near E, and if $F_1E$ is an incident ray, $EF_2$ will be the reflected ray.

**Paraboloidal Mirror.**—As the distance $F_2F_1$ between the foci of an ellipse increases, the ellipse approximates more and more closely to a parabola. Thus, if we wish to design a mirror which shall bring all rays from an infinitely distant object to a point focus (Fig. 25), we must give the section of the mirror the form of a parabola, while the surface of the mirror has the form of a paraboloid of revolution. Mirrors for astronomical telescopes are given this form.

FIG. 25.—Paraboloidal Mirror.

## Questions on Chapter II

1. Given the law of reflection, prove that the image of an object in a plane mirror is on the perpendicular to the mirror, and as far behind as the object is in front.

2. When a horizontal beam of light falls on a vertical plane mirror, which revolves about a vertical axis in its plane, show that the reflected beam revolves at twice the rate of the mirror.

3. Prove that when light falls directly on a concave spherical mirror of radius $r$, from a point at a distance $u$ from the mirror, then an image is formed at a distance $v$, where $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$.

4. If the mirror MM (Fig. 17, p. 26) is concave, while the telescope T is removed, and a small illuminated aperture is placed immediately below the scale CD, prove that a real image of the aperture will be formed on the scale, when the distance from the scale to the mirror is equal to twice the focal length of the latter.

If the mirror is deflected through an angle $\theta$, and the image is displaced through a distance $d$ cms. on the scale, prove that—

$$\theta = \frac{d}{2r},$$

where $r$ is the radius of curvature of the mirror in centimetres.

5. A real image is formed by a concave mirror, and this is (1) observed directly, and (2) thrown on a white screen. How will the brightness of the image seen in either case depend on the aperture of the mirror?

6. A real object is situated at the centre of curvature of a concave mirror. Show by a graphical construction that the image is inverted, and coincides in position with the object.

7. A virtual object is formed at the centre of curvature of a convex mirror. Show by a graphical construction that the image is inverted, and coincides in position with the object.

8. A small object is placed on the axis of a spherical mirror (convex or concave) at a distance U from the focus of the mirror. Prove that the resulting image is formed on the axis at a distance V from the focus, where $UV = f^2$; the focal length of the mirror being equal to $f$

# CHAPTER III

## APPLICATIONS OF THE LAWS OF REFRACTION

**Introductory.**—A statement has already been made (p. 8) of the laws governing the refraction of light at the interface between two different media. Before applying these laws it must be remarked that the value of the refractive index $\mu$ depends, not only on the nature of the media separated by the interface, but also on the colour of the refracted light. Thus, when blue light is transmitted from air into water, the index of refraction has a slightly greater value than would be found for yellow or red light. In the present chapter the dependence of the refractive index on the colour of the refracted light will be ignored. The results obtained will be strictly true for light of a definite colour, such as the yellow light emitted by a Bunsen flame into which some common salt has been introduced; they will be only approximately true for ordinary white light.

The laws of refraction may, by the aid of suitable apparatus, be verified by direct experiment. However, our confidence in the truth of these laws rests, not so much on the result of any single experiment, as on the perfect agreement between the theoretical deductions from them, and the results of accurate observations.

**Definitions.**—When light is refracted at a surface separating two different material media, the ratio of the sines of the angles of incidence and refraction will, in the following pages, be termed the **index of refraction** at the surface. When light is refracted from a vacuum (or, what is very nearly equivalent, from air) into a material medium, the ratio $\sin i/\sin r$ will be termed the **refractive index of the material medium.**

### Plane Refracting Surface

**Graphical Construction for Refracted Ray.**—Let AB (Fig. 26) represent the section of a surface, perpendicular to the plane of the paper, separating two different media. Let the index of refraction from the upper to the lower medium be equal to $\mu$, and let IC be a ray in the upper medium incident on AB in the plane of the paper. With C, the point of incidence, as centre, describe two circular arcs, DE and FH, of which the respective radii are in the ratio $\mu/1$. Through G, the point where the arc FH is cut by the incident ray, draw a line perpendicular to AB; and from K, the point where this perpendicular cuts the arc DE, draw the line KC to the point of incidence. Then KC, produced into the lower medium, gives the direction of the refracted ray.

FIG. 26.—Graphical Construction for Refracted Ray.

Let CN be the normal at C. Draw KL and GM perpendicular to CN. Then, sine of angle of incidence GCN = MG/CG, while sine of angle KCN = LK/CK. Since LK = MG, and, by construction, CK = $\mu \times$ CG, sin GCN = MG/CG = $\mu \times$ LK/CK = $\mu$ sin KCN. Thus, KCN is the angle of refraction, and KC produced is the refracted ray.

**Law of Refraction for Small Angles of Incidence.**—When light is incident normally on a refracting surface, $i = 0$, and $\sin i = 0$. In this case, since $\sin i = \mu \sin r$, we must have $\sin r = 0$, and $r = 0$, or the light is transmitted normally, without refraction. When the angle $i$ is small, $r$ must also be small, and we may, without sensible error, substitute the circular measures of the angles for the sines (p. 29). In this case—

$$i = \mu r,$$

a relation which we shall subsequently find useful.

**Reversibility of the Path of Light.**—Let a ray, IC (Fig. 26), incident at a point C on the interface between two different

media, give rise to a refracted ray CR in the lower medium. If we reverse the ray CR, as, for instance, by reflecting it normally from a plane mirror, experiment shows that the reversed ray RC gives rise to a refracted ray travelling along the path CI in the upper medium. This is an instance of the following general law: **If light, after suffering any number of reflections and refractions, has its final path reversed, the previous path of the light will be completely retraversed in a reversed direction.**

Let ${}_1\mu_2$ be the index of refraction from the upper to the lower medium (Fig. 26). Then, if the angle of incidence, ICN, is equal to $i$, while the angle of refraction, RCN′, is equal to $r$—

$$\sin i/\sin r = {}_1\mu_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

On reversing the ray CR, the new angle of incidence at C is equal to RCN′ $= r$, while the angle of refraction into the upper medium is equal to ICN $= i$. Then, if ${}_2\mu_1$ is the index of refraction from the lower to the upper medium—

$$\sin r/\sin i = {}_2\mu_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying together corresponding sides of (1) and (2), we obtain—

$$1 = {}_1\mu_2 \times {}_2\mu_1 ; \therefore {}_2\mu_1 = 1/{}_1\mu_2.$$

**Thus, the index of refraction from one medium** (B) **to another** (A) **is equal to the reciprocal of the index of refraction from** A **to** B.

**Refraction through a Plate.**—Let AB, EF (Fig. 27) be sections of the parallel bounding surfaces of a plate of glass or other transparent medium, of which the index of refraction is equal to $\mu$. The surfaces of which AB, EF are the sections, are supposed to be perpendicular to the plane of the paper. Let there be a vacuum above and below the plate. A ray, IC, incident in the plane of the paper at an angle $i$, gives rise to a refracted ray, CD, in the plane of the paper, since the normal at C is in the plane of the paper. Let $r$ be the angle of refraction at C; then—

Fig. 27.—Refraction through a Plate.

$$\sin i/\sin r = \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The ray CD is incident at an angle $r$ on the surface EF. The emergent ray DK lies in the plane of the paper; if it makes an angle $r'$ with the normal—

$$\sin r/\sin r' = 1/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Multiplying together corresponding sides of (3) and (4), we obtain—

$$\sin i/\sin r' = 1, \text{ or } \sin i = \sin r'.$$

Thus $r' = i$, and **the emergent ray DK is parallel to the incident ray IC.**

The ray DK is displaced laterally with respect to IC. Produce IC, and from D draw DM perpendicular to IC produced. Then the lateral displacement is equal to MD.

From C draw CN normal to AB, cutting EF in N. Let CN, the thickness of the plate, be equal to $t$. Then—

$$CN/CD = \cos r \text{ ; } \therefore \ CD = t/\cos r.$$

$$MD/CD = \sin DCM = \sin (i - r).$$

$$\therefore \ MD = CD \sin (i - r) = \frac{t \sin (i - r)}{\cos r}.$$

Thus, the lateral displacement of the ray is proportional to $t$, the thickness of the plate, and, by making $t$ sufficiently small, the displacement may be made as small as we please.

EXPT. 2.—Obtain a rectangular parallelopiped of glass, and place it on a sheet of drawing paper. Run a pencil along two opposite edges, AB and EF (Fig. 27), so as to mark the positions of these on the paper. Place two pins (upright) in any straight line, IC, on one side of the glass, and arrange two more pins in a straight line, DK, on the opposite side of the glass, so that on looking through the glass along the row of pins, all four appear to be in a straight line. Remove the glass, and draw pencil lines, IC and KD, through the positions of the pins, cutting the lines AB and EF in C and D. Show that IC is parallel to DK. Join CD, and draw the normals at C and D. Measure the angles of incidence and refraction at C and D, and calculate the refractive index of the glass.

**Refraction through a Compound Plate.**—Let ICDEF (Fig. 28) be the path of a ray which successively traverses two parallel plates with a common interface AB. Experiment shows that the emergent ray EF is parallel to the incident ray IC, provided that the same medium extends above and below the compound plate.

Let there be a vacuum above and below the compound plate. If $\mu_1$ is the refractive index of the upper plate while the angle of incidence at C is equal to $i$, then the ray CD enters the upper plate at an angle, $r_1$, determined by the equation—

$$\sin i/\sin r_1 = \mu_1. \quad . \quad . \quad (5)$$

The angle of incidence of the ray CD on the interface AB, is equal to $r_1$. If the angle of refraction into the lower plate is equal to $r_2$, and if ${}_1\mu_2$ is the index of refraction from the upper to the lower plate, then—

FIG. 28.—Refraction through a Compound Plate.

$$\sin r_1/\sin r_2 = {}_1\mu_2. \quad . \quad . \quad (6)$$

The ray DE is incident at an angle $r_2$ on the lowest face, and since the ray EF emerges into a vacuum at an angle $i$, we have—

$$\sin r_2/\sin i = 1/\mu_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $\mu_2$ is the refractive index of the lower plate. Multiplying together corresponding sides of (5) and (7), and using (6), we obtain—

$$\frac{\sin r_2}{\sin r_1} = \frac{\mu_1}{\mu_2} = 1/{}_1\mu_2.$$

$$\therefore \ {}_1\mu_2 = \mu_2/\mu_1.$$

Thus, **the index of refraction from one medium** (A) **to another** (B) **is equal to the refractive index of** B, **divided by that of** A.

The index of refraction from a vacuum to ordinary atmospheric air is equal to 1·0003. Thus, if $\mu$ is the index of refraction from a vacuum to glass, the index of refraction from air to glass is equal to $\mu/1{\cdot}0003$, a value scarcely differing from $\mu$.

Let light be successively refracted at $n$ parallel, plane surfaces separating $(n + 1)$ media, of which the first and last are identical, while the rest differ. Then the angle of incidence, $i$, at the first surface is equal to the angle of emergence from the last surface. Let ${}_1\mu_2$, ${}_2\mu_3$, ${}_3\mu_4$, . . . ${}_m\mu_n$, ${}_n\mu_1$, be the respective indices of refraction at the 1st, 2nd, 3rd, . . . $(n - 1)$th, and $n$th surfaces. Then, if the angles of refraction at the

1st, 2nd, 3rd, . . . $(n-1)$th, and $n$th surfaces are respectively equal to $r_1, r_2, r_3, \ldots r_l, r_m, i$, we have—

$$\begin{aligned}
\sin i/\sin r_1 &= {}_1\mu_2.\\
\sin r_1/\sin r_2 &= {}_2\mu_3.\\
\sin r_2/\sin r_3 &= {}_3\mu_4.\\
\ldots \quad \ldots &= \ldots\\
\sin r_l/\sin r_m &= {}_m\mu_n.\\
\sin r_m/\sin i &= {}_n\mu_1.
\end{aligned}$$

Multiplying all of these equations together, we obtain—

$$\begin{aligned}
&{}_1\mu_2 \,.\, {}_2\mu_3 \,.\, {}_3\mu_4 \,.\, \ldots \,.\, {}_m\mu_n \,.\, {}_n\mu_1\\
&= \frac{\sin i \,.\, \sin r_1 \,.\, \sin r_2 \,.\, \ldots \,.\, \sin r_l \,.\, \sin r_m}{\sin r_1 \,.\, \sin r_2 \,.\, \ldots \,.\, \sin r_m \,.\, \sin i} = 1.
\end{aligned}$$

**Total Internal Reflection.**—When light is refracted from a dense to a rarer medium, the index of refraction, $\mu'$, is less than unity. If $i$ is the angle of incidence of a ray in the denser medium, then the ray refracted into the rarer medium is inclined at an angle $r$ to the normal, given by the equation—

$$\sin r = \sin i/\mu'. \qquad \ldots \ldots \ldots \qquad (8)$$

Now, the sine of 90° is equal to unity, and no angle can be found with a sine greater than unity. Thus, for (8) to lead to a real value of $r$, we must have $\sin i/\mu'$ equal to, or less than, unity. Consequently, for a refracted ray to be formed, the greatest possible value of $i$ is given by the equation—

$$\sin i = \mu'.$$

This value of $i$ is termed the **critical angle.** For angles of incidence exceeding the critical value, no light is refracted through the surface, all of it being internally reflected.

FIG. 29.—Total Internal Reflection.

The paths of a number of rays from a luminous point, situated beneath the surface of a dense medium, are shown in Fig. 29. The ray OB, incident at the critical angle, gives rise to a refracted ray which travels along the surface of the medium

($r = 90°$). If ON is normal to the surface, the angle BON is the critical angle. Rays incident at an angle greater than BON are totally reflected internally.

Let the index of refraction of the denser medium be equal to $\mu$, while the refracting surface separates the medium from a vacuum, or air. Then in (7), $\mu' = 1/\mu$, and the critical value of $i$ is given by—

$$\sin i = 1/\mu.$$

For crown glass, $\mu = 1 \cdot 5$. Thus, the critical angle is one of which the sine is equal to $1/1 \cdot 5 = 0 \cdot 666$. This angle is equal to $42°$ (nearly). For water ($\mu = 1 \cdot 33$) the critical angle is equal to $49°$ (nearly).

Expt. 3.—Blacken a metal ball in a smoky flame, and lower it into a beaker of water. While immersed in the water the ball appears as if its surface were of polished silver. A thin air film is enclosed between the water and the smoked surface, and all rays falling on this film, at angles greater than $49°$, are totally reflected.

Let the surface of a medium A, of refractive index $\mu_1$, be covered by a parallel layer of a medium B, of refractive index $\mu_2$; the free surface of the medium B being in contact with the air. Then it can easily be proved that if light is incident on the interface between A and B, at an angle exceeding the critical value for the medium A, then the light refracted into B will fall on the free surface of B at an angle greater than the critical angle for B, and will thus be totally reflected.

Let a ray fall on the interface between A and B at an angle of incidence, $i$, equal to the critical value for medium A. Then, $\sin i = 1/\mu_1$. The ray refracted into medium B is inclined to the normal at an angle $r$, given by the equation—

$$\sin i/\sin r = \mu_2/\mu_1. \quad \therefore \sin r = \sin i. \quad \mu_1/\mu_2 = 1/\mu_2.$$

Since $r$ is the angle of incidence on the free surface of medium B, it is seen that the light refracted into B falls on the free surface at the critical angle. It is easily seen that an increase of $i$ leads to an increase of $r$, so that if $i$ exceeds the critical value for medium A, $r$ will exceed the critical value for medium B.

The above result can be utilised in a method of experimentally determining the refractive index, in terms of the critical angle, of a liquid.

EXPT. 4.—A wooden box, open at one end, is provided with two narrow vertical slits in opposite sides. The closed end of the box is bored normally to receive a rod provided with a pointer; the latter rotates above a scale of degrees pasted on the end of the box. Inside the box a cell is attached to the rod. This cell is made from two pieces of plate glass separated by narrow strips of paper laid between opposite edges, the space between the plates being sealed off from its surroundings by means of sealing-wax or bicycle cement. The cell is attached to the rod with its plane faces parallel to the latter.

A rectangular vessel, with plate-glass sides, is filled with the liquid of which the refractive index is required, and the wooden box is inverted over it, the cell being immersed in the liquid. Adjustment is made so that, on looking through one of the slits, the other slit is seen through the liquid and the cell immersed therein. It is best to place a Bunsen flame, into which some common salt has been introduced, on an iron or platinum wire, opposite the slit to be viewed. Rotate the cell till the slit illuminated by yellow light just disappears, the light being totally reflected from the air film. Observe the position of the pointer, and then rotate the cell in the opposite direction till the slit again disappears. The angle through which the cell is rotated between the two disappearances is equal to twice the critical angle of the liquid for the yellow light of the sodium flame.

**Totally Reflecting Prisms.**—In optics, the term *prism* is generally applied to a body bounded by three planes which intersect in three parallel straight lines. A plane perpendicular to these lines of intersection cuts the prism in a triangular section, termed a **principal section** of the prism; the angles of the triangle are termed the **angles of the prism.** For the moment we need only consider a prism of which the angles are equal to 45°, 45°, and 90°.

A ray of light, after entering such a prism normally through one of the mutually rectangular faces, will fall on the hypotenuse face at an angle of 45°, which is greater than the critical angle for glass. It is, therefore, totally reflected, and emerges normally from the remaining face of the prism. In this manner the direction of a ray may be deflected through an angle of 90° with only a small loss of intensity.

In the optical projection of apparatus on a screen, an inverted image is obtained. Where inversion is disadvantageous, an appliance termed an **erecting prism** may be employed to produce an erect image. An erecting prism is merely a glass prism with angles of 45°, 45°, and 90°. A ray, incident on one of the mutually rectangular faces in a plane coinciding with a principal section, and in a direction parallel to the

hypotenuse face, is refracted so as to be incident on the hypotenuse face at an angle exceeding the critical angle. After reflection the ray emerges from the third face parallel to its original direction. The manner in which inversion is produced can be readily understood from Fig. 30.

FIG. 30.—Erecting Prism.

**Image formed by Refraction at a Plane Surface.**—Let AB (Fig. 31) be the section of a plane surface, perpendicular to the plane of the paper, separating two media of different optical densities. Let the media below and above the surface AB possess refractive indices respectively equal to $\mu_1$ and $\mu_2$; in Fig. 31, $\mu_2 > \mu_1$. Let O be a luminous point in the plane of the paper, at a perpendicular distance CO $(= u)$ below the surface. Let a ray OD from O be incident, in the plane of the paper, at a small angle $i$. Since OC is normal to the surface, the angle DOC is equal to $i$. The refracted ray DE, corresponding to the incident ray OD, must lie in the plane of the paper. Produce ED backwards to cut CO (produced) in I, and let CI $= v$. Then, if the refracted ray is inclined to the normal at an angle $r$, this angle is equal to DIC. Also (p. 46), since the angle of incidence, $i$, is small—

$$i = \frac{\mu_2}{\mu_1} r.$$

Further, since we may measure the angles $i$ and $r$ by their tangents—

$$\frac{CD}{u} = \frac{\mu_2}{\mu_1}\frac{CD}{v}; \quad \therefore \quad v = \frac{\mu_2}{\mu_1} u.$$

FIG. 31.—Image formed by Refraction at a Plane Surface.

Now, this relation between $v$ and $u$ is independent of the exact position of the incident ray OD, provided that the angle of incidence is small. Thus, all rays from O, which are incident on the surface at small angles, must

diverge from a virtual focus, I, after refraction at the surface, so that I is the image of O.

If the medium surrounding the object is optically rarer than that on the other side of the refracting surface, $\mu_2/\mu_1$ is greater than unity, and $v$ is greater than $u$, as in Fig. 31. If $\mu_2 < \mu_1$, $v$ is less than $u$, or the image is nearer to the surface than the object. When an object, immersed in water of refractive index equal to $\mu$, is viewed normally to the free surface, we must write $\mu_1 = \mu$, $\mu_2 = 1$. Then—

$$u/v = \mu.$$

This gives us a means of determining the refractive index of a liquid.

EXPT. 5.—Obtain a cylindrical glass vessel, about 30 or 40 cms. in height, and place a small fragment of chalk at its bottom to serve as an object. Fill the vessel with water, and measure the actual distance, $u$, of the chalk from the water surface. On looking down into the water the chalk appears to be raised above its true position. To obtain the position of the image of the chalk, place a small, horizontal, pointed gas flame above the surface, and adjust till there is no parallax (p. 23) between the image of the flame reflected in the water and the refracted image of the chalk. A piece of glass tubing, drawn out to a fine point, may be used as a burner for the flame. The distance from the flame to the surface of the water is numerically equal to $v$. Calculate the value of the refractive index of the water. This will be found to be equal to 1·33, or 4/3.

Thus, **an object situated below the surface of water appears, when viewed normally to the surface, to be at three-quarters of its real distance below the surface.**

EXPT. 6.—Place a fine-pointed needle at the bottom of a small beaker, and view this by means of a low-power microscope[1] which can be raised or lowered through measured distances. Read the position of the microscope, then fill the beaker with a liquid, again focus the microscope on the needle, and obtain a reading. Finally, sprinkle a few fragments of cork dust on the surface of the water, focus the microscope on one of these, and obtain a third reading. The differences between the first and third readings gives the value of $u$, while that between the

[1] A travelling microscope, suitable for this and many other experiments, is made by Mr. Wilson, Belmont Road, Chalk Farm, N.W. This microscope can also be adjusted for use as a horizontal or vertical kathetometer, and is invaluable in a physical laboratory.

second and third readings gives $v$. Calculate the refractive index of the liquid.

The refractive index of a thick plate of glass, on opposite surfaces of which ink marks have been made, may be found in a similar manner.

When an object below the surface of water is viewed in a direction considerably inclined to the normal to the surface, the apparent distance of the object will not be that obtained above. In Fig. 29, if we produce any two neighbouring refracted rays backwards, their point of intersection gives the position of the image formed by the pencil bounded by those rays. Thus, it becomes evident that, as the direction of vision becomes more and more inclined to the normal, the image rises to a greater height within the water.

**Image formed by Refraction through a Transparent Plate.**—Let an object be viewed through a transparent plate with parallel faces, placed perpendicular to the line of vision: Let $\mu$ be the refractive index, and $t$ the thickness of the plate. If $u$ is the distance of the object from the surface of the plate opposed to it, then the image formed by refraction at that surface will be at a distance $(\mu u)$ from the latter (p. 53). This image serves as a virtual object with respect to the refraction at the second surface, and since the distance of this virtual object from the latter is equal to $(\mu u+t)$, and refraction occurs from the plate into air, the distance of the corresponding image from the second surface of the plate is equal to—

$$(\mu u + t)/\mu = (u + t/\mu).$$

The distance of this image from the first surface is equal to $(u+t/\mu-t)$, so that, when seen through the plate, the object appears to be nearer than it really is, by a distance equal to $t(\mu-1)/\mu$.

**Refraction through a Prism.**—Let ABK (Fig. 32) be the principal section of a prism of a transparent medium, supposed to be surrounded by air. Let us consider the refraction of a ray of light which enters the prism by one face, AB, and emerges from another face, AK, without having suffered internal reflection. The edge A of the prism is termed the **refracting edge**, and the angle KAB is termed the **refracting angle** of the prism. Let LC be a ray incident on AB in the plane of the paper, at an angle $i_1$. If $\mu$ is the index of refraction of the

medium composing the prism, the refracted ray CD is inclined to the normal at an angle $r_1$, determined by—

$$\sin i_1/\sin r_1 = \mu.$$

Let the ray CD be incident on the face AK at an angle $r_2$; then the ray DE refracted into the air lies in the plane of the paper, and makes an angle $i_2$ with the normal, in accordance with the equation—

$$\sin i_2/\sin r_2 = \mu.$$

FIG. 32.—Refraction through a Prism.

Thus, the rays LC, CD, DE, all lie in the plane of the paper. Produce the incident ray LC to F, and produce the emergent ray DE backwards to meet CF in G. Then the angle FGE, which will be denoted by $\delta$, measures the deviation which has been produced in the ray LC by refraction through the prism; this angle is termed the **angle of deviation.**

From Fig. 32, it is easily seen that $\angle$GCD $= i_1 - r_1$, while $\angle$GDC $= i_2 - r_2$. Then, since FGD is the external angle of the triangle GCD, and is therefore equal to the sum of the internal opposite angles GCD and GDC—

$$\delta = i_1 - r_1 + i_2 - r_2.$$

**Angle of Minimum Deviation.**—When $\mu$ and $i_1$, together with the refracting angle of the prism, are known, $r_1$, $i_2$, and $r_2$ can be calculated, so that the angle of deviation, $\delta$, becomes known. The deviation varies with the value of $i_1$, and experiment shows that, with a given prism, there is a certain value of $i_1$ for which the angle of deviation has a minimum value. The smallest value which, for a given prism, the deviation can have, is termed the **angle of minimum deviation.**

EXPT. 7.—Rule a straight line across a sheet of paper mounted on a drawing board. Fasten one of the triangular faces of a prism to the flat head of a drawing pin, make a small vertical hole in the drawing board through a point on the ruled line, and insert the shank of the drawing pin. The prism can now be rotated freely. Place two common pins upright at different points in the ruled line, on one side

of the prism. On looking through the prism in a suitable direction these two pins can be seen. Place two more pins upright in the drawing board, on the same side of the prism as the eye, in such positions that these, together with the pins seen through the prism, appear to be in a straight line. We thus obtain the positions of an incident, and the corresponding emergent, ray. Notice that, in passing through the prism, the ray is deviated *away from* the refracting edge. Rotate the prism through a small angle, again obtain the position of the emergent ray, and note the change in the angle of deviation. Repeat this procedure, rotating the prism through small angles in the same direction. Observe that for one particular position of the prism the deviation is less than for any other position. Mark the positions of the incident and emergent rays for the position of minimum deviation, run a pencil point round the sides of the prism, and after removing the latter, mark the direction of the ray inside the prism.

It can now be proved that, **for a ray to suffer minimum deviation, the angle of incidence, $i_1$, and that of emergence, $i_2$, must be equal, and the ray** CD (Fig. 33) **within the prism must be equally inclined to the two faces.**

For let the angles $i_1$ and $i_2$ (Fig. 32) be unequal when the ray LC suffers minimum deviation. Reverse the emergent ray DE; then the ray ED, incident at an angle $i_2$, gives rise to the ray CL, emerging at an angle $i_1$ (p. 47); the deviation is the same as before, and must therefore have a minimum value. Thus, the ray LC, incident at an angle $i_1$, suffers minimum deviation, as does the ray ED, incident at an angle $i_2$, and there must consequently be *two* angles of incidence which lead to minimum deviation, which is contrary to experience. Thus, $i_1 = i_2$. In this case $r_1 = r_2$, and $\angle$ ACD = $\angle$ ADC (Fig. 33).

**Determination of Refractive Index.** — When, with respect to a given prism, the refracting angle $a =$ KAB (Fig. 33) is known, and the angle of minimum deviation $\delta =$ FGE has been observed, the refractive index of the substance composing the prism can be calculated.

FIG. 33.—Angle of Minimum Deviation.

For, let $i_1 = i_2 = i$, while $r_1 = r_2 = r$. Then (p. 56)—

$$\delta = 2(i - r); \therefore i = (\delta + 2r)/2.$$

Also, DAC $= \alpha$, ACD $=$ ADC $= \left(\frac{\pi}{2} - r\right)$. Then, since the three angles of a triangle are together equal to two right angles—

$$\angle \text{DAC} + \angle \text{ACD} + \angle \text{ADC} = \pi, \text{ or } \alpha + 2\{(\pi/2) - r\} = \pi.$$
$$\therefore r = \alpha/2.$$
$$i = (\delta + 2r)/2 = (\delta + \alpha)/2.$$
$$\therefore \mu = \frac{\sin i}{\sin r} = \frac{\sin\{(\alpha + \delta)/2\}}{\sin(\alpha/2)}.$$

Expt. 8.—From the results of Expt. 7, calculate the refractive index of the prism supplied to you.

**Deviation produced by Acute-Angled Prism.**—Let light be incident at a small angle on a prism with a very small refracting angle, $\alpha$.

From p. 56,

$$\delta = i_1 + i_2 - (r_1 + r_2).$$

From Fig. 32, since $\angle$ DAC + $\angle$ ACD + $\angle$ ADC $= \pi$,

$$\alpha + \{(\pi/2) - r_1\} + \{(\pi/2) - r_2\} = \pi.$$
$$\therefore r_1 + r_2 = \alpha.$$

Then, since $i_1 = \mu r_1$ (p. 46), and $i_2 = \mu r_2$,

$$\delta = (\mu - 1)(r_1 + r_2) = (\mu - 1)\alpha.$$

**Image formed by Refraction through a Prism.**—When a ray is refracted through a prism, so that its angles of incidence and emergence are equal, the deviation of the ray is a minimum. It follows from this that if a narrow divergent pencil is refracted through a prism, so that the axial ray of the pencil follows the path of minimum deviation, then the emergent pencil diverges from a virtual point focus, and is similar to the incident pencil. For the angles of incidence of the extreme rays of the pencil differ but slightly from the angle of incidence of the axial ray, and the deviation changes but slowly with the angle of incidence when the latter is nearly equal to that corresponding to minimum deviation. Similarly, when crossing a valley, on reaching the lowest point, or point of minimum height, we travel, for a short distance, in a horizontal straight line, without either ascending or descending.

**Thus, an object, seen through a prism, is most distinct when the axial ray of the pencil reaching the eye has followed the path of minimum**

deviation (Fig. 34). For other paths the extreme rays of the pencil are deviated by different amounts, the ray on one side of the pencil being deviated more, and that on the opposite side of the pencil less, than the axial ray, and on emergence, the rays do not diverge from any single point. This result will subsequently be found useful.

FIG. 34.—Image formed by Refraction through a Prism.

### SPHERICAL REFRACTING SURFACE

**Definitions.**—The terms used in connection with a spherical refracting surface are similar to those applied to a mirror. The terms *centre* and *radius of curvature*, *principal section*, *aperture*, *pole*, and *principal axis* have the meanings defined on p. 27. The position of a point on the principal axis is defined by its distance from the pole; distances measured from the pole in a direction opposite to that of the incident light are positive; those measured from the pole in the direction of the incident rays are negative. A concave refracting surface has a positive radius of curvature, while a convex refracting surface has a negative radius of curvature.

In order to completely specify a spherical refracting surface, the index of refraction at the surface must be given, in addition to the data necessary to define the surface as a mirror.

**Refraction at Concave Surface.**—Let APB (Fig. 35) be a principal section of a concave surface, separating a vacuum (or air), on the right, from a medium of refractive index equal to $\mu$, on the left. Let C be the centre of curvature, P the pole, and OP the principal axis of the surface. Let O be a luminous point on the axis. Let a ray from O be incident at E in the plane of the paper. The radius CEN, drawn through E, forms the normal to the surface at E. Then the refracted ray EF also lies in the plane of the paper, and if produced backwards, will cut the axis at a point I.

Let $\angle OEC = i$, while $\angle FEN = \angle IEC = r$. When $i$ is small, $i = \mu r$. Then $\angle OEI = (i - r) = (\mu - 1)r$.

Let $\angle$ ECP = C, while $\angle$ EIP = I, and $\angle$ EOP = O. Then—

$$\angle \text{ECP} = \angle \text{OEC} + \angle \text{EOC}, \quad \text{or} \quad C = \mu r + O. \quad . \quad . \quad (1)$$

Also—

$$\angle \text{EIP} = \angle \text{OEI} + \angle \text{EOC}, \quad \text{or} \quad I = (\mu - 1)r + O. \quad . \quad (2)$$

FIG. 35.—Incident and Refracted Rays, Concave Surface.

Multiply (1) throughout by $(\mu - 1)$, and (2) by $\mu$, and subtract. Then—

$$\mu I - (\mu - 1)C = O \; ; \; \therefore \; \mu I - O = (\mu - 1)C.$$

When the angles O, C, and I are small, they may be measured by their tangents (p. 29). Draw ED perpendicular to the axis, and let DE $= y$. Let PO (= DO nearly) $= u$, while PI (= DI nearly) $= v$, and PC (= DC nearly) $= r_1$. Then, reasoning as on p. 31,

$$(\mu y/v) - (y/u) = (\mu - 1)y/r_1.$$

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since this equation is independent of $y$, all rays from O, inclined at a small angle to the axis, will form a refracted pencil virtually diverging from a point I. Thus, I is the image of O, and the relation between the distances of the object and its image is given by (3).

**Refraction at a Convex Surface.**—Let APB be a principal section of a convex surface, separating a vacuum (or air), on the right, from a medium of refractive index equal to $\mu$, on the left. Let C be the centre of curvature, P the pole, and OPC the principal axis, of the surface. Let O be a luminous point on the axis. Let a ray from O be incident at E in the plane of the paper. Draw the radius CEN through E. Then, since CEN, a line in the plane of the paper, forms the normal to the

surface at E, the refracted ray EI lies in the plane of the paper; let it cut the axis at I, a point on the negative side of the surface.

FIG. 36.—Incident and Refracted Rays, Convex Surface.

Produce IE to F. Let $\angle$ OEN $= i$, while $\angle$ IEC = FEN $= r$. When $i$ is small, $i = \mu r$. Then $\angle$ OEF $= (i - r) = (\mu - 1)r$.

Let $\angle$ EOP = O, while $\angle$ ECP = C, and EIP = I. All of these angles are supposed to have positive values. Then—

$$\angle \text{OEN} = \angle \text{ECP} + \angle \text{EOP}, \quad \text{or} \quad \mu r = \text{C} + \text{O}. \quad . \quad . \quad (4)$$

$$\angle \text{OEF} = \angle \text{EIP} + \angle \text{EOP}, \quad \text{or} \quad (\mu - 1)r = \text{I} + \text{O}. \quad . \quad (5)$$

Multiply (4) throughout by $(\mu - 1)$, and (5) by $\mu$, and subtract. Then—

$$\mu\text{I} - (\mu - 1)\text{C} + \text{O} = 0; \quad \therefore \mu\text{I} + \text{O} = (\mu - 1)\text{C}.$$

Draw ED perpendicular to the axis, and let DE $= y$. When the angles O, I, and C are small, they may be measured by their tangents. Let PO (= DO nearly) $= u$, while PI (= DI nearly) $= v$, and PC (= DC nearly) $= r_1$. Then, since I and C must have positive values, while $v$ and $r_1$ (in Fig. 36), are negative, I $= -y/v$, and C $= -y/r_1$. Also, O $= y/u$. Then—

$$-\mu y/v + y/u = -(\mu - 1)y/r_1.$$

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equations (3) and (6) are identical, so that a single equation may be used for both concave and convex surfaces, due regard being paid to signs. We may now drop the subscript number, added to $r$ in order to avoid confusion between the angle of refraction at E and the radius of

the spherical surface; and the general formula connecting the distances of image and object from the pole may be written—

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Since the refracted ray always occupies a position on the negative side of the surface, it is plain that, for the image I to be real (*i.e.* for light to *actually pass* through the point I), $v$ must be negative. When $v$ is positive, the image I is virtual. When light is refracted from a medium of refractive index $\mu_1$, to one of refractive index $\mu_2$, we must substitute $\mu_2/\mu_1$ for $\mu$ in (7).

**Conjugate Foci.**—Light from a point O (Fig. 36), in the medium to the right of the refracting surface, is brought to a real focus I at a point in the medium on the left of the surface. Since the path of the rays may be reversed, a luminous point at I, in the medium to the left of the refracting surface, will form an image O on the right of the surface. Thus, O and I are **conjugate foci.**

In Fig. 35, a divergent pencil from O, after entering the medium to the left of the refracting surface, appears to diverge from the virtual focus I; hence a pencil in the medium to the left of the surface, converging toward the virtual object I, will, after refraction into the medium to the right of the surface, converge toward the real image O.

**Principal Foci.**—There are two points on the axis of a refracting surface which merit special attention. In equation (7) above, let $v = \infty$; then $1/v = 0$; further, the rays after refraction at the surface are parallel to the axis. Substituting $1/v = 0$ in (7), we find that—

$$1/u = -(\mu - 1)/r; \therefore u = -r/(\mu - 1).$$

This value of $u$ is termed the **First Principal Focal Distance** of the refracting surface; it may be denoted by $f_1$. A point at a distance $f_1 = -r/(\mu - 1)$ from the pole of the surface, is termed the **First Principal Focus** of the surface. **An incident ray proceeding from the first principal focus** ($f_1$ positive), **or toward that point** ($f_1$ negative), **gives rise to a refracted ray parallel to the axis.**

In equation (7), let $u = \infty$; then $1/u = 0$; further, the incident rays diverge from an infinitely distant point on the axis, so

that they are parallel to the axis. Substituting $1/u = 0$ in (7), we find that—

$$\mu/v = (\mu - 1)r\,;\ \therefore\ v = \mu r/(\mu - 1).$$

This value of $v$ is termed the **Second Principal Focal Distance** of the refracting surface ; it may be denoted by $f_2$. A point at a distance $f_2 = \mu r/(\mu - 1)$ from the pole is termed the **Second Principal Focus** of the surface. **An incident ray parallel to the axis gives rise to a refracted ray proceeding from the second principal focus** ($f_2$ positive), **or toward that point** ($f_2$ negative).

It is easily seen that—

$$\mu f_1 + f_2 = 0.$$

If the media on the positive and negative sides of the surface have refractive indices respectively equal to $\mu_1$ and $\mu_2$, then we must substitute $\mu_2/\mu_1$ for $\mu$ in the above results, when we find that—

$$f_1/\mu_1 + f_2/\mu_2 = 0.$$

The two principal focal distances of a surface always have opposite signs. For a concave surface ($r$ positive), the first principal focal distance is negative, while the second principal focal distance is positive. For a convex surface ($r$ negative) the first and second principal focal distances are respectively positive and negative.

**Relative Positions of Object and Image.**—(1) *Concave Surface.*—When the object is at the centre of curvature, rays from it fall normally on the surface, and refraction does not occur. In this case image and object coincide. When $\mu$ is greater than unity, the second principal focal distance, $f_2$, is positive. Substituting in (7) (p. 62), we find—

$$\mu/v - 1/u = \mu/f_2\,;\ \therefore\ \mu/v = 1/u + \mu/f_2.$$

Thus, when $u$ is positive (real object), $v$ is positive, and the image is virtual. It can easily be proved, from a diagram similar to Fig. 35, that when $u$ is less than $r$, $v$ is also less than $r$; when $u$ is greater than $r$, $v$ is also greater than $r$. In both cases the image is nearer to the centre of curvature than the object.

(2) *Convex Surface.*—In this case $r$ is negative; and the second principal focal distance, $f_2$, is negative, while the first principal focal distance, $f_1$, is positive. Then—

$$\mu/v = 1/u - 1/f_1.$$

When $u$ is greater than $f_1$, $v$ is negative, and a real image is formed on the negative side of the refracting surface. When $u$ is less than $f_1$, $v$ is positive, and the image is virtual.

**Object of Finite Dimensions.**—Let LPM (Fig. 37) be the axis of a concave refracting surface of which the pole is at P and the centre of curvature at C. Let the medium on the left of the surface possess a refractive index $\mu$, while a vacuum (or air) is to the right of the surface. Let OA be a small object, of which one end, O, is on the axis, while OA is perpendicular to the axis. The image of the point O will be formed at a point I on the axis. To find the image of the point A, we must determine the intersection of two refracted rays, initially derived from that point.

Fig. 37.—Graphical Construction for Image.

Through P, the pole of the surface, draw a plane perpendicular to the axis. This plane, termed the **principal plane**, has properties similar to the principal plane of a mirror (p. 36).

Let $PF_2 = f_2 = \mu r/(\mu - 1)$, where $r$ is the radius of curvature of the surface. Then $F_2$ is the second principal focus. Similarly, if $PF_1 = f_1 = -r/(\mu - 1)$, $F_1$ is the first principal focus. Since the surface is concave, $r$ is positive, and the focal points have the positions shown. The following rays may now be determined from the results previously obtained :—

1. The ray AD, parallel to the axis, gives rise to a refracted ray, virtually proceeding from the second principal focus $F_2$ (p. 63).

2. The ray AE, directed toward the first principal focus, gives rise to a refracted ray parallel to the axis (p. 62). Produce this refracted ray backwards.

3. The ray ACK, passing through the centre of curvature C, is transmitted at the surface without refraction.

4. The axis PM is normal to the surface at P. Thus, a ray AP, incident at a small angle APO, gives rise to a refracted ray inclined at an angle BPO to the axis, where—

$$\angle \text{APO} = \mu \times \angle \text{BPO}.$$

The point of intersection, B, of any two of the lines DB, EB, PB, and KB, gives the image of the point A. It can be proved that

ect in a single point. A line BI, drawn from

the axis, gives the image of OA.

for a convex surface, separating a medium

, on the left, from a vacuum (or air) on the

construction employed.

If there is a vacuum (or air) to the left, and a medium of refractive index $\mu$ to the right, of

Fig. 38.—Graphical Construction for Image.

the surface, the object being in the latter medium, we must substitute $1/\mu$ for $\mu$ in (7) (p. 47). Thus, $f_1 = r\mu/(\mu - 1)$, and $f_2 = -r/(\mu - 1)$.

As will be proved in Chap. VII, the optical system of the eye is approximately equivalent to a chamber filled with a medium of refractive index $\mu = 4/3$, and provided with a convex window of which the radius of curvature $r$ is equal to $-5$ mm. In this case $f_1 = 15$ mm., and $f_2 = -20$ mm. A construction similar to that used in Fig. 38 may be employed to find the ocular image of an object placed in front of the eye. The object will naturally be placed beyond the first focal point. In that case the ray AE must be drawn so as actually to pass through $F_1$.

**Magnification.**—In Fig. 37, let OA = PD = o, while IB = PE = i. Then the magnification produced by refraction at the surface is equal to i/o. For the meaning of a negative sign in connection with magnification, see p. 38.

Let PO = $u$, while PI = $v$. Then—

1. From the similar triangles $DPF_2$, $BIF_2$,

$$IB/PD = F_2I/F_2P = (PF_2 - PI)/PF_2\,;\ \therefore\ i/o = (f_2 - v)/f_2\ . \quad (8)$$

2. From the similar triangles $EPF_1$, $AOF_1$,

$$PE/OA = F_1P/F_1O = -PF_1/(PO - PF_1)\,;\ \therefore\ i/o = -f_1/(u - f_1)\ . \quad (9)$$

3. From the similar triangles AOC, BIC,

$$IB/OA = CI/CO = (PI - PC)/(PO - PC)\,;\ \therefore\ i/o = (v - r)/(u - r) \quad (10)$$

4. Since the angles APO and BPI are supposed to be small, they may be measured by their tangents. Thus—

$$\mu(IB/PI) = OA/PO\,;\ \therefore\ i/o = v/\mu u.\ \ .\ .\ . \quad (11)$$

Thus, collecting results, we find that—

$$\frac{i}{o} = -(v - f_2)/f_2 = -f_1/(u - f_1) = (v - r)/(u - r) = v/\mu u.$$

Using equation (7) (p. 62), together with the values of $f_1$ and $f_2$, the student should find no difficulty in proving that these expressions for the magnification are equal (compare p. 39), and thus proving that the various construction lines in Fig. 37 intersect in a single point B. It forms a useful exercise to determine the magnification from Fig. 38.

From (11) it follows that when $v$ is negative the image is real (p. 62), and the magnification is negative; then the image is inverted. When $v$ is positive, the image is virtual and erect.

## LENSES

**Definitions.**—A lens is a portion of a refracting medium bounded by two curved surfaces, generally spherical in contour. One of the surfaces may be plane, in which case it may be considered to form a small part of a spherical surface of infinite radius; just as a small portion of the earth's surface is sensibly plane, though it forms part of a very large sphere. A line drawn through the centres of curvature of the two surfaces of the lens is termed the **principal axis** of the lens. When one surface is plane, the principal axis is normal to that surface, and passes through the centre of curvature of the other surface. The points where this axis cuts the surfaces of the lens, are termed the **poles** of those surfaces. In the present chapter it will be

supposed that the lens is thin; *i. e.*, the two poles are supposed to be so close together that distances may be measured from either indifferently. Distances measured from the lens, in a direction opposite to that of the incident light, are positive; those measured in the reverse direction are negative. When the peripheral boundary of the lens is circular, the diameter of the boundary is termed the **aperture** of the lens. A section of the lens through the principal axis, is termed a **principal section.**

The forms of the principal sections of a number of characteristic lenses are shown in Fig. 39. Of these, the lenses A, B, and C decrease in thickness toward the periphery. A is termed **double-convex,**

FIG. 39.—Sections of Typical Lenses.

or **bi-convex.** B is termed **plano-convex,** or **convexo-plane,** according as the plane or the convex surface faces the incident rays. C is termed **convexo-concave,** or **concavo-convex.** The lenses D, E, F increase in thickness toward the periphery. D is termed **double-concave,** or **bi-concave,** E is termed **plano-concave,** or **concavo-plane,** and F is termed **convexo-concave,** or **concavo-convex.**

**Refraction through a Lens.**—Let a lens be formed of a medium of refractive index $\mu$, and be placed in a vacuum (or in air). Let a luminous point be situated on the axis at a distance $u$ from the nearer surface of the lens. Then, if $r_1$ is the radius of curvature of this surface, and $v'$ is the distance of the image formed by refraction thereat, we have (p. 62)—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $t$ be the distance between the poles of the two surfaces of the lens. Then the image formed by refraction at the first

surface will be at a distance $(v'+t)$ from the pole of the second surface. Let the radius of curvature of the second surface be equal to $r_2$. Since the second refraction takes place in passing from the medium composing the lens to a vacuum (or air), the index of refraction at the second surface is equal to $1/\mu$. Then, if the image formed by the second refraction is at a distance $v$ from the pole of the second surface, we have—

$$\frac{(1/\mu)}{v} - \frac{1}{v'+t} = \frac{(1/\mu)-1}{r_2}; \therefore \frac{1}{v} - \frac{\mu}{v'+t} = \frac{1-\mu}{r_2}. \quad . \quad (2)$$

If the lens is thin, $t$ may be neglected in comparison with $v'$. Then—

$$\frac{1}{v} - \frac{\mu}{v'} = \frac{1-\mu}{r_2}. \quad \ldots\ldots\ldots \quad (3)$$

If we add (1) to (3), we eliminate $v'$. Then we obtain—

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots\ldots \quad (4)$$

Thus, **light from a point on the axis, at a distance $u$ from the lens, forms an image at a distance $v$ from the lens.** When $v$ is positive, the image is virtual; when $v$ is negative, the rays actually pass through the image, and the latter is real.

**Principal Foci.**—Let $v = \infty$, so that $1/v = 0$ in (4). Then the refracted rays are parallel to the axis, and the corresponding position of the object is given by—

$$\frac{1}{u} = -(\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots\ldots \quad (5)$$

The value of $u$ determined from (5) is termed the **First Principal Focal Distance** of the lens; this may be denoted by $f_1$. Then a point on the axis, at a distance $f_1$ from the lens, is termed the **First Principal Focus** of the lens. **A ray proceeding from the first principal focus** ($f_1$ positive), **or toward that point** ($f_1$ negative), **is rendered parallel to the axis after refraction through the lens.**

Let $u = \infty$, so that $1/u = 0$ in (4). Then the incident rays are parallel to the axis, and the corresponding position of the image is given by—

$$\frac{1}{v} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots\ldots\ldots \quad (6)$$

The value of $v$ derived from (6) is termed the **Second Principal Focal Distance** of the lens; this may be denoted by $f_2$. Then a point on the axis at a distance $f_2$ from the lens is termed the **Second Principal Focus** of the lens. **An incident ray parallel to the axis gives rise to a refracted ray which virtually proceeds from the second principal focus** ($f_2$ positive), **or which actually passes through that point** ($f_2$ negative).

It is obvious from (5) and (6) that the two focal lengths of a lens (in air) are equal in magnitude, but opposite in signs. Thus, the two focal points are on opposite sides of the lens, and are equidistant from it. It is convenient to speak of **the Focal Length of a lens, meaning thereby the second principal focal distance of the lens.** Thus, if $f$ is the focal length of a lens—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Then (4) may be re-written—

$$1/v - 1/u = 1/f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

**Focal Lengths of Characteristic Lenses.**—1. *Bi-Convex Lens.*—The radius of curvature of the surface facing the incident rays is negative, that of the remaining face being positive (Fig. 39). If R and S are the numerical magnitudes of $r_1$ and $r_2$, then $r_1 = -$ R, and $r_2 = +$ S. Consequently—

$$\frac{1}{f} = -(\mu - 1)\left(\frac{1}{R} + \frac{1}{S}\right).$$

Since $\mu$ is supposed to be greater than unity, we see that the focal length of a bi-convex lens is negative.

If we turn the lens end for end, $r_1 = -$ S, and $r_2 =$ R. In this case it is easily seen that the focal length is the same as before.

2. *Plano-Convex Lens.*—If the plane surface faces the incident rays, $r_1 = \infty$, and $1/r_1 = 0$. The radius of curvature of the second surface must be positive. Let $r_2 = +$ S. Then—

$$1/f = (\mu - 1)\{-(1/S)\} = -(\mu - 1)(1/S).$$

Thus, the focal length of a plano-convex lens is negative. It is easily seen that, if the lens is turned end for end, the focal length remains the same as before.

3. *Concavo-Convex Lens.*—Here the centres of curvature of both surfaces are on the positive side of the lens, so that $r_1$ and $r_2$ are both

positive. If $r_1$ is *greater* than $r_2$, $1/r_1$ is less than $1/r_2$, and the focal length is negative. In this case, represented by C (Fig. 39), the thickness of the lens decreases toward the periphery. Turning the lens end for end leaves the focal length unaffected.

If $r_1$ is *less* than $r_2$ (both being positive), $1/r_1$ is greater than $1/r_2$, and the focal length is positive. In this case, represented by F (Fig. 39), the thickness of the lens increases toward the periphery.

4. *Bi-Concave Lens.*—In this case the radius of curvature of the surface facing the incident rays is positive ($= + R$, say), while that of the remaining surface is negative ($= - S$, say). Then—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} + \frac{1}{S}\right),$$

and the focal length is positive. The thickness of this lens increases toward the periphery (Fig. 39, D).

5. *Plano-Concave Lens.*—Here $r_1 = \infty$, and $r_2$ is negative ($= - S$, say). Then—

$$1/f = (\mu - 1)(1/S),$$

and the focal length is positive. The thickness increases toward the periphery.

Thus **lenses which increase in thickness toward the periphery have positive focal lengths, while those which decrease in thickness toward the periphery have negative focal lengths.**

After refraction through a lens of negative focal length, rays, initially parallel to the axis, converge to a real focus; such lenses are termed **convergent.** After refraction through a lens of positive focal length, rays, initially parallel to the axis, diverge from a virtual focus; such lenses are termed **divergent.**

**Relative Positions of Image and Object.**—1. *Divergent Lens.*—From the equation—

$$1/v = 1/u + 1/f,$$

it is evident that if $f$ is positive (divergent lens), a positive value of $u$ gives a positive value of $v$. Thus, a real object produces a virtual image. Also, since the value of $1/f$ is added to that of $1/u$, to give the value of $1/v$, the value of $1/v$ must be greater than that of $1/u$, and $v$ is less than $u$. Thus, **the image is nearer to the lens than the object.**

2. *Convergent Lens.*—Let $f$ be negative, and numerically equal to F. Then—

$$1/v = 1/u - 1/F.$$

If $1/u$ is less than $1/F$ (*i.e.* if $u$ is greater than F, and the object is farther from the lens than the first principal focus), $1/v$ is negative, and therefore $v$ is negative, and the image is real.

If $1/u$ is greater than $1/F$ (*i.e.* if $u$ is less than F, and the object is nearer to the lens than the first principal focus), $1/v$ is positive, and therefore $v$ is positive, and the image is virtual.

**Object of Finite Dimensions.**—Let LM (Fig. 40) be the axis of a lens, of which the poles are situated on opposite sides of the point P. If the lens is thin, both poles may be considered to be situated at P. Through P, draw a plane GPK, perpendicular to the axis. This plane may be termed the **principal plane** of the lens; its properties are similar to those of the principal plane of a mirror or refracting surface (pp. 36 and 64). Let $F_1$ and $F_2$ be the first and second principal foci of the lens; in Fig. 40, the position of these foci correspond to a divergent lens. Let OA be a small object, of which one extremity, O, is on the axis, while OA is perpendicular to the axis. To find the image of the point A we have the following construction:—

FIG. 40.—Graphical Construction for Image.

1. A ray AD, incident parallel to the axis, gives rise to a refracted ray virtually proceeding from $F_2$, the second principal focus (p. 69).

2. An incident ray AE, directed toward the first principal focus $F_1$, gives rise to a refracted ray parallel to the axis (p. 68). Produce this latter ray backwards.

3. The axis is normal to both surfaces of the lens at the poles. Consequently, the two sides of the lens are mutually parallel in

the immediate neighbourhood of the poles, and since the lens is thin, a ray AP, incident at P, is undeflected, just as if it passed through a very thin plate of glass.

The point of intersection, B, of any two of the lines DB, EB, and PB, gives the image of A. A line BI, drawn from B perpendicular to the axis, gives the complete image of OA.

Fig. 41 shows the construction when the lens is convergent.

FIG. 41.—Graphical Construction for Image.

**Magnification.**—Let OA = PD = **o** (Fig. 40), while IB = PE = **i**. Then the magnification produced by the lens is measured by the ratio **i**/**o**. When this ratio is negative, the image is inverted (p. 38); otherwise the image is erect. Let PO = $u$, while PI = $v$. Let $f_1 = -f$, and $f_2 = +f$. Then $f$ is the focal length (p. 69) of the lens. Thus, in Fig. 40, $PF_2 = f$, while $PF_1 = -f$.

1. From the similar triangles BIP, AOP,

$$IB/OA = PI/PO\ ;\ \therefore\ \mathbf{i}/\mathbf{o} = v/u. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

2. From the similar triangles $EPF_1$, $AOF_1$,

$$PE/OA = F_1P/F_1O = -PF_1/(PO - PF_1)\ ;\ \therefore\ \mathbf{i}/\mathbf{o} = +f/(u+f) \quad . \quad (9)$$

3. From the similar triangles $BIF_2$, $DPF_2$,

$$IB/PD = IF_2/PF_2 = (PF_2 - PI)/PF_2\ ;\ \therefore\ \mathbf{i}/\mathbf{o} = -(v-f)/f \quad . \quad (10)$$

Thus—

$$\mathbf{i}/\mathbf{o} = v/u = f/(u+f) = -(v-f)/f. \quad . \quad . \quad . \quad . \quad (11)$$

By the aid of the equation—

$$1/v - 1/u = 1/f,$$

it is easily shown that (8), (9), and (10) are equivalent, which proves that the lines DB, EB, and PB intersect in a single point.

**Two or more Lenses in Contact.**—Let two thin lenses, of focal lengths equal to $f_1$ and $f_2$, be adjusted so that both have a common axis. Then, if a luminous point on the axis, at a distance $u$ from the first lens (that of focal length $f_1$), gives rise, by refraction through the first lens, to an image at a distance $v'$ from the latter, we have—

$$1/v' - 1/u = 1/f_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let $t$ be the distance between the two thin lenses. Then the image formed by the first lens is at a distance $(v' + t)$ from the second lens; and, as this image takes the place of object with respect to the second lens, we may determine the distance $v$ of the final image from the second lens from the equation—

$$1/v - 1/(v' + t) = 1/f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the distance between the lenses is very small (as, for instance, when the lenses are in contact), we may neglect $t$, when (13) becomes—

$$1/v - 1/v' = 1/f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Adding (12) and (14), we eliminate $v'$, and obtain—

$$1/v - 1/u = 1/f_1 + 1/f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

When the incident rays are parallel to the axis, $1/u = 0$. Let F be the corresponding value of $v$ in (15); then F is the **second principal focal distance** (or **the focal length**) of the lens combination. From (15)—

$$1/F = 1/f_1 + 1/f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In words, **two thin lenses, of focal lengths $f_1$ and $f_2$, when placed in contact, are equivalent to a single lens of focal length F, determined from (16). The single lens of focal length F, when placed in the position occupied by the lens combination, produces an image of a given object, in the same position and of the same size as that produced by the combination.**

It is easily proved, by extending the reasoning already used, that a number of thin lenses, of focal lengths respectively equal to $f_1$, $f_2$, $f_3$, . . . $f_n$, are jointly equivalent, when placed in contact one with another, to a single lens of focal length F, determined by—

$$1/F = 1/f_1 + 1/f_2 + 1/f_3 + \ . \ . \ . + 1/f_n. \quad . \quad . \quad . \quad (17)$$

In using this formula it must be remembered that $f_1, f_2, f_3, \ldots f_n$ are the *second principal focal lengths* of the respective lenses, and due care must be taken to give these their appropriate signs.

**Power of a Lens.**—Equation (17) shows that when a number of lenses are in contact one with another the algebraical sum of the reciprocals of their respective focal lengths is equal to the reciprocal of the focal length of the equivalent lens. This result suggests that, for purposes of calculation, it is convenient to deal with the reciprocals of the focal lengths rather than with the focal lengths themselves. **The reciprocal of the focal length of a lens is termed the power, or dioptric strength, of that lens.** Ophthalmic surgeons use a unit of power termed the **dioptre.** This is the power of a lens of 1 metre focal length. It is further agreed that the power of a convergent lens shall be positive, while that of a divergent lens shall be negative. Thus, **to find the power, in dioptres, of a given lens, express the focal length in terms of the metre, obtain its reciprocal, and change the sign of the result.** The power of a combination of lenses in contact is equal to the algebraical sum of the powers of the constituent lenses.

**Two Lenses separated by a Finite Distance.**—When two lenses, arranged so as to have a common axis, are separated by a distance too great to be neglected, it is impossible to find a single thin lens which, when placed in any fixed position, shall produce an image of the same size, and in the same position, as that produced by the combination. But a single thin lens can be found, which, when placed at a suitable fixed point, produces an image *of the same size, but not generally in the same position*, as that produced by the combination. This lens is said to be **equivalent** (in the restricted sense defined above) to the combination.

Let $P_1Q$, $P_2R$ (Fig. 42), be the respective principal planes of two lenses, arranged so as to have a common axis LM, the distance $P_2P_1$ between them being equal to $d$. Let OA be an object perpendicular to the axis; it is required to find the position and focal length of a single thin lens, which shall produce an image of OA of the same size as that formed by refraction through the lenses $P_1Q$ and $P_2R$, whatever may be the position of OA. Fig. 42 is drawn on the supposition that the lenses $P_1Q$ and $P_2R$ are both divergent.

Let AB be a particular incident ray from A, meeting the axis, when produced, in a point E. Let the direction of this ray be such that, after refraction through $P_1Q$, it follows the path BC, and finally, after refraction through $P_2R$, travels along CD parallel to the axis. Let BC produced cut the axis in L. Produce CD backwards to G; then it is obvious that the size of the final image is determined by the distance of the ray CD from the axis. To determine the image, let the incident ray AS be directed toward the first principal focus of $P_1Q$; the corresponding refracted ray ST emerges from $P_1Q$ parallel to the

FIG. 42.—Graphical Determination of a Lens, equivalent to a Combination of Two Lenses.

axis, and after refraction at $P_2R$, virtually proceeds from U, the second principal focus of $P_2R$. Then, the point of intersection $A'$ of the lines CG and TU gives the final image of A, and a line from this point, drawn perpendicular to the axis, gives the complete image $I_1A'$.

Let the lines CG and BE intersect in H, and through H draw HK perpendicular to the axis. Let us remove the lenses $P_1Q$ and $P_2R$, and substitute in their stead a thin lens with pole at K and principal focus at E, its principal plane passing through HK, perpendicular to the axis. Then the incident ray AH gives rise to the refracted ray HD parallel to the axis. Draw AK through the pole of the lens HK. Then the point $A''$, where the lines AK and CG intersect, gives the image of A formed by refraction through the lens HK, and the line $A''I_2$, drawn perpendicular to the axis, is the image of AO. It is obvious that $A''I_2 = A'I_1$, and the lens HK, of which the first principal focus is at E, is equivalent to the lens combination $P_1Q$ and $P_2R$.

The distance EK, which is equal to the focal length of the equivalent lens, can be found as follows. The triangles $BP_1E$, HKE, are similar, and $HK = CP_2$. Also the triangles $BP_1L$, $CP_2L$, are similar. Then—

$$EK/EP_1 = HK/BP_1 = CP_2/BP_1 = LP_2/LP_1.$$

$$\therefore\ EK = (EP_1 \times LP_2)/LP_1.$$

Further, since the ray BC, directed toward L, emerges from the lens $P_2R$ parallel to the axis, L must be the *first principal focus* of $P_2R$, and $LP_2 = -P_2L =$ the focal length ($f_2$ say) of the lens $P_2R$. Also the points L and E are conjugate with respect to the lens $P_1Q$. Let the lens $P_1Q$ have a positive focal length equal to $f_1$. Then—

$$1/P_1L - 1/P_1E = 1/f_1.$$

$$\therefore\ EP_1 = -P_1E = (f_1 \times LP_1)/(f_1 + LP_1).$$

$$\therefore\ EK = \frac{EP_1 \times LP_2}{LP_1} = \frac{f_1 \times LP_2}{f_1 + LP_1} = \frac{f_1f_2}{f_1 + f_2 + d},$$

since $LP_1 = LP_2 + P_2P_1 = f_1 + d$. Thus, if the focal length of the equivalent lens is equal to F (= EK),

$$\frac{1}{F} = \frac{f_2 + f_1 + d}{f_1f_2} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{d}{f_1f_2}. \quad . \quad . \quad . \quad . \quad (18)$$

Equation (18) has been obtained from Fig. 42, which is drawn on the supposition that $f_1$ and $f_2$ are both positive. It will furnish an instructive exercise for the student to prove that (18) also holds when either $f_1$ or $f_2$ is negative, or when both are negative.

Let $P_1K = a$. In Fig. 42, $a$ is negative; when $a$ is positive, the equivalent lens must be placed in front of the first lens $P_1Q$ of the combination. Then, as proved above—

$$EK/EP_1 = LP_2/LP_1.$$

$$\therefore\ \frac{EK}{EK - P_1K} = \frac{F}{F - a} = \frac{f_2}{f_2 + d}.$$

$$\therefore\ 1 - \frac{a}{F} = 1 + \frac{d}{f_2}, \text{ and } a = -d\frac{F}{f_2}.$$

Thus, **the equivalent lens must be placed at a distance ($-dF/f_2$) in front of, or at a distance ($+dF/f_2$) behind, the first lens.**

**Aplanatic Surface and Foci.**—The formula expressing the relation between the distances of an object and its image from a lens has been obtained on the supposition that the rays from

the object, and also those which form the image, are only slightly inclined to the axis. When these conditions are not complied with, the pencil of rays from the object gives rise, by refraction through the lens, to a bundle of rays which have no common point of intersection. Thus, the laws deduced in the present chapter apply only to a lens of small aperture; they are departed from, to a considerable extent, when the aperture is large, and this departure is termed **spherical aberration.** It is impossible to design a lens of wide aperture, which shall bring all of the rays falling on it from *any* point on the axis to a point focus on the axis; but a lens may be constructed which accurately brings all rays falling on it, from a *particular* point on the axis, to a point focus on the axis. Such a lens is said to be **aplanatic,** and the particular positions of the object and image are termed **aplanatic foci.**

Let APB (Fig. 43) be a principal section of a hemispherical surface, with centre at C and pole at P, the line PC produced being the axis. Let the surface separate a vacuum (or air) on the left from a medium of refractive index $\mu$ on the right. Let $r$ be the radius of curvature of the surface, and let O be a luminous point, on the axis, at a distance CO $= r/\mu$ from the centre of curvature. Let OD be any ray from O, in the plane of the paper, falling on the surface at D, and let DE be the corresponding ray refracted through the surface. Since DE is in the plane of the paper, produce it backwards to cut the axis in the point I. Then it may be proved that CI $= \mu r$, and **all rays from O, after refraction at the surface, diverge from the single point I.**

FIG. 43.—Aplanatic Foci of Spherical Refracting Surface.

From Snell's law—

$$\sin \mathrm{IDC}/\sin \mathrm{ODC} = \mu.$$

Since any two sides of a triangle are in the same ratio as the sines of the opposite angles, and CD = $r$,

$$\frac{\text{CD}}{\text{CO}} = \frac{r}{r/\mu} = \mu = \frac{\sin \text{DOC}}{\sin \text{ODC}}.$$

$$\therefore \sin \text{DOC} = \sin \text{IDC, and } \angle \text{ DOC} = \angle \text{ IDC}.$$

Further, $\angle$ DOC = $\angle$ IDO + $\angle$ DIO; $\therefore$ $\angle$ IDO + $\angle$ DIO = $\angle$ IDC = IDO + ODC. $\therefore$ $\angle$ DIO = ODC.

From the triangle IDC,

$$\text{CI/CD} (= \text{CI}/r) = \sin \text{IDC}/\sin \text{DIO} = \sin \text{IDC}/\sin \text{ODC} = \mu ;$$

$$\therefore \text{CI} = \mu r.$$

Since this result is independent of the angle which the incident ray CD makes with the axis, it follows that all rays from the point O are refracted so as to virtually proceed from the point I. Thus O and I are the aplanatic foci of the surface APB.

The above result is utilised in the construction of very high power microscope objectives. The object to be viewed is immersed in cedar-wood oil, and the lowest lens of the microscope objective consists of a glass hemisphere, the plane face of which is turned toward the object, and is immersed in the oil. Since the refractive index $\mu$ of cedar-wood oil is equal to that of the glass, the above conditions will be complied with if the object is placed at a distance $r/\mu$ from the plane face of the lens, $r$ being the radius of the hemisphere. It is easily seen that the magnification produced is equal to $\mu^2$. This arrangement is generally termed **Abbe's homogeneous immersion**, from its inventor. It possesses other advantages which will be explained in the chapter on diffraction.

PROBLEM.—A lens is placed in front of a small illuminated aperture in a white screen. Find the condition that the light internally reflected from the back surface of the lens, shall form an image on the screen close to the illuminated aperture.

Let $\mu$ be the refractive index of the lens, and let the illuminated aperture be at a distance $u$ from its first surface, of which the radius of curvature is equal to $r_1$. Then the distance $v'$ of the image formed by refraction at the first surface is given (p. 62) by—

$$\mu/v' = 1/u + (\mu - 1)/r_1.$$

If the lens is thin, and $v'$ is equal to $r_2$, the radius of curvature of the second surface, then all rays will be incident normally on the second surface, and their paths will be reversed. In this case an image of the aperture is formed on the screen. Then—

$$\mu/r_2 = (\mu - 1)/r_2 + 1/r_2 = 1/u + (\mu - 1)/r_1.$$
$$\therefore\ 1/r_2 = 1/u + (\mu - 1)\{1/r_1 - 1/r_2\} = 1/u + 1/f,$$

where $f$ is the focal length of the lens.

Then—

$$r_2 = \frac{uf}{u+f}.$$

**Intrinsic Luminosities of Image and Object.**—Let an object of area A be situated in a medium of refractive index $\mu_1$, at a distance $u$ from a surface on the negative side of which is a medium of refractive index $\mu_2$. If the image formed by refraction at the surface is at a distance $v$ from the latter, the linear dimensions of image and object will be in the ratio $\mu_1 v/\mu_2 u$ (p. 66), and the area of the image will be equal to $\mu_1^2 v^2 A/\mu_2^2 u^2$. If L is the intrinsic luminosity of the object, and A′ is the area of the refracting surface, the rate at which light falls on the latter is equal to $AA'L/u^2$, and if the whole of the light falling on the surface goes to form the image, the intrinsic luminosity of the latter is equal to—

$$AA'L/u^2 \div \{(A'/v^2) \times (\mu_1^2 v^2 A/\mu_2^2 u^2)\} = \mu_2^2 L/\mu_1^2;$$

(compare p. 40).

If the light is now refracted, at a second surface, into a medium of refractive index $\mu_3$, the intrinsic luminosity of the new image will be equal to—

$$\frac{\mu_3^2}{\mu_2^2} \times \frac{\mu_2^2 L}{\mu_1^2} = \frac{\mu_3^2}{\mu_1^2} L.$$

Thus, if, after refractions at any number of surfaces, the light finally enters a medium of refractive index $\mu_n$, the intrinsic luminosity of the final image will be equal to—

$$\mu_n^2 L/\mu_1^2.$$

If the object is in air, and light from it, after any number of refractions, finally emerges into air, then $\mu_n = \mu_1$, and **the intrinsic luminosities of image and object are equal.**

As a matter of fact, some light is lost by reflection at each of the refracting surfaces; and, since no medium is absolutely

transparent, still more light is lost during transmission through the various media. Consequently, **an image formed by refraction through a system of lenses is always less bright than the object;** the greater the number of lenses, the less bright will be the image. A further decrease of brightness will occur if the pencils from the final image do not fill the pupil of the eye (p. 41). Provided that the pencils from the final image fill the pupil, the brightness of the image is independent of the aperture of the lens. On the other hand, owing to the occurrence of spherical aberration in an ordinary lens of wide aperture, the distinctness with which details in the image can be seen will be greatest when the aperture of the lens is moderately small.

When a lens is used to form an image on a screen, the brightness of this image will be increased by increasing the aperture of the lens, since here we have to deal with the total amount of light falling on unit area of the screen.

A telescope used by night is found to render distant objects brighter than when these are seen by the unaided eye. This appears to be due to the circumstance that small objects cannot be seen distinctly under feeble illumination, owing to the physiological properties of the eye, while larger objects are clearly visible

## Questions on Chapter III

1. State the two laws of refraction, and explain how both of them are required, and can be used, to determine the relation between the apparent and real depth of water when viewed perpendicularly to its surface.

2. Explain the apparent raising of a picture stuck at the bottom of a cube of glass, till it appears to an eye looking down as if it were in the glass. If the index of refraction is 1·5, how much does the picture appear raised to perpendicular vision?

3. You are given a drawing board, paper, and drawing materials, also some pins, and a rectangular block of glass with polished faces. How would you proceed to verify the law of refraction, and to determine the refractive index of the glass?

4. Explain and describe the effect of atmospheric refraction on the apparent positions of the heavenly bodies.

5. Show how to calculate the distance by which an object appears to be shifted when a piece of glass, bounded by parallel plane surfaces, is interposed squarely across the direction between you and the object.

How may this apparent change of distance be used to measure the refractive index of the glass? Why do not distant objects appear to be shifted?

6. Draw the paths of a number of rays proceeding from any one point of a horizontal object under water, and indicate the apparent positions of three distinct points of the object to an eye above the water.

7. Show that, when light passes through a thin prism, the deviation is very approximately constant whatever the angle of incidence, provided the incidence is nearly perpendicular.

Show that, in the same circumstances, the deviation of the portion of the light that is reflected back from the second face of the prism differs from that of the light reflected back from the first face by a constant amount.

8. Show how to use the phenomenon of total internal reflection, in a practical manner, to measure the refractive index of a liquid.

9. What is a total reflection prism? Explain with a sketch how such a prism can be used to deflect a beam of light into a direction at 90° from its original course. What angles must be given to such a prism in order that it may deflect a beam through 60°?

10. A block of transparent jelly of refractive index 1·33 is bounded on one side by part of the convex surface of a sphere of radius 8 mm. Find the position of the principal focus within the mass of the material.

11. A glass globe, 12 inches in diameter, is filled with water ($\mu = 4/3$). Trace the changes in position of the image, seen by an observer looking along a diameter, of a point in the water as it moves from the farther to the nearer end of the diameter. Neglect the thickness of the glass.

12. Prove the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ for refraction through a concave lens, and show that the image formed by such a lens is virtual, erect, and diminished.

13. Show how to determine, either graphically or arithmetically, the position and magnitude of the image of an object placed in front of a convex lens. An arrow 5 inches long is placed 8 inches away from a convex lens whose focal length is 3 inches. Find the position and length of the image.

14. An incandescent gaslight, with a mantle 10 centimetres high, stands at the same level as a converging lens, the power of which is + 5 dioptres, situated 6 metres to the right of the light. Find the position and size of the image of the mantle. If the light is then lifted up 1 metre above its former position, what change will take place in the position of the image?

15. If the refractive index from air to glass is $\frac{3}{2}$, and that from air to water is $\frac{4}{3}$, find the ratio of the focal lengths of a glass lens in water and in air.

16. A person with a window behind him, on looking into a convex lens, sees two images of the window. Describe the character and mode of formation of these images.

17. What is meant by the aperture of a lens? What has the aperture of a lens to do with (*a*) the definition, and (*b*) the brightness, of the images which the lens is used to form?

18. Show that it is impossible for the image formed by a train of lenses to be brighter than the object. What are the principal causes which may reduce the actual brightness observed by the eye below this ideal limit?

19. A convex and a concave lens, each 10 inches in focal length, are held coaxially at a distance of 3 inches apart. Find the position of the image, if the object is at a distance of 15 inches beyond (*a*) the convex, (*b*) the concave lens.

20. Two lenses, one a positive, the other a negative lens, are placed upon a common principal axis at a distance apart. Find the conditions in order that a parallel beam of rays entering the system by one lens, shall emerge from the other lens as a parallel beam.

## Practical

1. Determine the refractive index of a plate of glass by means of a microscope.

2. Find the refractive index of a plate of glass by observing the thickness, and the shift in a beam of light to which the glass is inclined.

3. Plot the path of a ray of light through the two given glass prisms.

4. Measure the refractive index of the given glass cube by plotting the path of a ray of light through it, and then measure sines of angles by means of a scale.

5. Arrange the two lenses given you so as to produce a real image of the object magnified four times, and show how to test your result.

6. Determine by experiment the form of a curve showing the relation between the distance of an object from a given lens, and the magnification of the real image.

# CHAPTER IV

## DISPERSION AND CHROMATIC ABERRATION

**Composite Character of Sunlight.**—When a beam of sunlight, admitted through a small aperture into a darkened chamber, is allowed to fall on a white screen, an image of the sun is depicted on the latter. The manner in which this image is formed has been described in connection with the pin-hole camera (p. 5). If the screen, instead of being white, has been painted a bright vermilion colour, the image of the sun is still formed, but it is now of a bright red colour. Similarly, on a green screen the image of the sun is green, and on a blue screen it is blue.

We have already learnt (p. 58) that, if a prism is interposed in the path of a narrow pencil of light, the pencil is deviated. If the light is homogeneous, and if the axial ray of the pencil follows the path of minimum deviation through the prism, then the incident and transmitted pencils are similar, and both diverge from points. Consequently, if sunlight be homogeneous, a beam admitted through a small aperture into a darkened chamber should merely be deviated when a prism is placed in its path, and a circular image of the sun should still be formed on a white screen. Experiment shows that this is not the case. The beam transmitted by the prism paints on the screen an elongated image, which is brilliantly coloured. One end of the image is red, and as we pass from this to the opposite end of the image, the colour changes through orange, yellow, green, and blue, to violet. This coloured image is termed a **spectrum**. The breadth of the image is equal to that of the image of the sun formed when the prism is removed. Moreover, if the screen

is painted a bright red colour, the light transmitted by the prism paints a red image of the sun on the screen, in the position occupied by the red part of the image when the screen was white. The explanation of these phenomena is, that sunlight is not homogeneous, but consists of numerous constituents which are deviated by different amounts when transmitted through a prism. When sunlight falls on a red screen, all of these constituents, with the exception of those which form the red end of the spectrum, are absorbed; the red constituents are diffusively reflected, and, on reaching the eye, give rise to a red image of the screen. In accordance with this theory, the colour of a body, when seen in sunlight, is due to a property of the body by which it absorbs some of the coloured constituents of sunlight; the remaining constituents are diffusively reflected, and form the ocular image of the body.

**Formation of a Spectrum.**—When a beam of sunlight is transmitted through a prism, the resulting spectrum is due to

FIG. 44.—Analysis of White Light by the aid of Crossed Prisms.

innumerable images of the sun, each being formed by a particular constituent of sunlight. It is found that the violet end of the spectrum is the more remote from the refracting edge of the prism, so that violet light is deviated to a greater extent than red light. Newton confirmed this conclusion by transmitting sunlight successively through two prisms, arranged with their refracting edges at right angles to each other. In Fig. 44, let VR be the spectrum formed by transmission through the prism nearer to the aperture S; V is the violet, and R the red, end of the spectrum. On placing the second prism in position, a

second spectrum, V'R', was formed, and the violet end V' was displaced from its previous position V, to a greater extent than the red end R' was displaced from its previous position R. Since, for a prism of given angle, the deviation depends merely on the refractive index of the prism (p. 58), it follows that **the refractive index is greater for violet than for red light.** This is often expressed by saying that the refrangibility of light increases from the red to the violet end of the spectrum. The variation of the refractive index of a substance with the colour of the transmitted light is termed **dispersion.**

The spectrum formed on a screen, in the manner described above, is not *pure*; *i.e.*, any particular point in the spectrum is not illuminated merely by one constituent of sunlight. This follows from the circumstance that each constituent of sunlight forms an image of the sun of finite dimensions, and the various images, formed by different constituents, overlap to a greater or less extent. Thus, in Fig. 45, the red image of the sun occupies the position RR, while the violet image occupies the position VV, and between these two images are those corresponding to the remaining constituents of sunlight. If the screen is removed, and the light transmitted by the prism is allowed to fall on the eye, each constituent of the sunlight diverges from a separate virtual image of the illuminated aperture. The violet image will appear at V' (Fig. 45), and the red image at R'. Thus, a **virtual spectrum** will be seen extending between V' and R'. The violet end of the spectrum appears nearer to the refracting edge of the prism. If the mean path of the rays corresponds nearly to that of minimum deviation through the prism, each coloured image of the small aperture will approximate to a point, and as overlapping will not, in this case, sensibly occur, the virtual spectrum will be pure.

FIG. 45.—Impure Spectrum formed on Screen.

**Formation of Pure Spectrum.**—If a lens be placed between the prism and the screen, as in Fig. 46, the virtual spectrum V'R' takes the place of an object, and a real image of this can be formed by the lens. In this case the rays corresponding to

each particular coloured constituent of sunlight, are focussed at a particular point on the screen, and the resulting spectrum is pure.

FIG. 46.—Formation of Pure Spectrum.

A similar result may be obtained by placing the lens between the aperture and the prism, as shown in Fig. 47. In this and the previous case, it is necessary that the mean path of the rays should correspond as nearly as possible with that of minimum deviation through the prism.

A third method of obtaining a pure spectrum will be described in connection with the spectrometer.

If the aperture has the form of a small circular hole, the spectrum will take the form of a narrow luminous line, red at one end and violet at the other. If, however, the aperture has the form of a narrow slit, of which the length is parallel to the refracting edge of the prism, then the spectrum will take the form of a luminous band of finite width.

FIG. 47.—Formation of Pure Spectrum.

**The Spectrometer.**—For the examination and measurement of spectra an instrument termed a **spectrometer** is used. The essential parts of this instrument comprise a collimator, SL (Fig. 48); a turn-table supporting a prism ABC; and a telescope, ME. The **collimator** consists of a metal tube, closed at one end

by a lens, L, and directed toward the axis of rotation of the turn-table. In the focal plane of the lens L is a vertical slit, S. Light diverging from this slit is rendered parallel (or collimated) by L, and a parallel pencil falls on the prism. Since parallel rays of homogeneous light are deviated to the same extent when refracted through a prism, it follows that each coloured constituent of the transmitted light will form a parallel pencil, which is deviated to an extent depending on its colour. Each pencil, after refraction through the telescope lens M, will be brought to a focus in the focal plane of M; thus a real image, VR, of the spectrum is formed in the focal plane of the telescope. This image is viewed through an eye-piece, E. If fine cross-wires are placed in the focal plane of M, any particular part of the spectrum can be brought into coincidence with their intersection. The telescope rotates so that it is always directed toward the axis of rotation of the turn-table, and is provided with a vernier which moves over a circular scale concentric with the turn-table.

FIG. 48.—Plan of Spectrometer.

The deviation corresponding to any particular part of the spectrum can be measured by setting the cross-wires on that part of the spectrum, reading the vernier attached to the telescope, and then removing the prism, setting the cross-wires on the image of the slit seen directly, and again reading the telescope vernier. The difference between these two readings gives the deviation.

When the collimator is properly adjusted, it is not necessary, for the formation of a pure spectrum, that the rays should traverse the prism along the path of minimum deviation.

Fig. 49 represents a spectrometer designed for the use of students, by Mr. Wilson, of 1, Belmont Street, Chalk Farm, N.W. It is a very serviceable instrument. The turn-table and telescope are provided with verniers reading to half a minute of arc. The turn-table can be raised or lowered, and is provided with levelling screws.

**Adjustment of Spectrometer.**—The following adjustments are necessary before commencing an experiment with the spectrometer.

1. To ADJUST THE EYE-PIECE.—The lens system of the eye-piece is movable in the tube which carries the cross-wires, and this tube can also be moved as a whole. Turn the telescope toward the sky, and move the eye-piece lenses till the cross-wires are distinctly seen.

2. To ADJUST THE TELESCOPE.—Direct the telescope toward a distant object (the top of a distant telegraph pole will serve), and move the tube carrying the eye-piece and cross-wires till there is no parallax between the image of the distant object and the cross-wires. **The cross-wires are then in the focal plane of the telescope.**

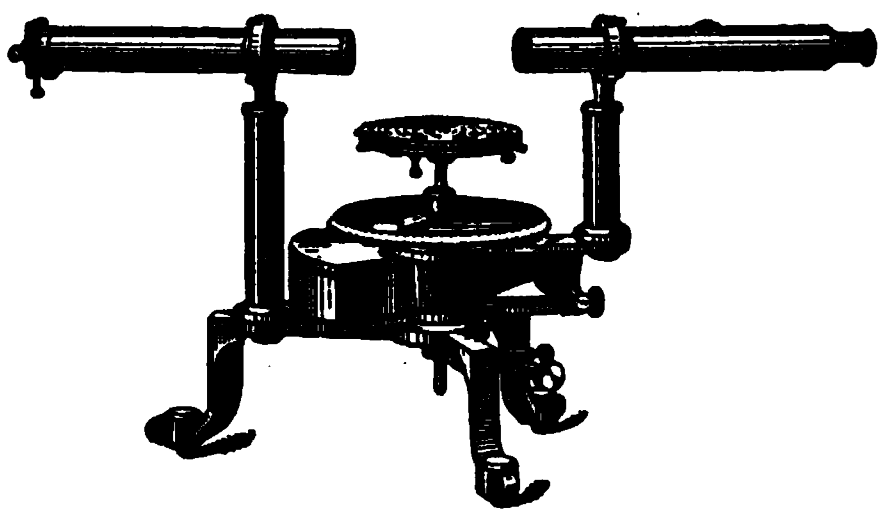


FIG. 49.—Spectrometer for Students.

3. To ADJUST THE COLLIMATOR.—Place a luminous flame in front of the slit, adjust the telescope so that it and the collimator tube are in a straight line, and then, while looking through the telescope, move the slit in or out till there is no parallax between its image and the cross-wires. **The slit is then in the focal plane of the collimator lens.** Adjust the slit to be vertical.

After the above adjustments have once been made, if there is any difficulty in seeing the cross-wires, the eye-piece lenses may be moved, but *not* the cross-wires themselves.

**Measurement of the Angles of a Prism.**—An optical arrangement for measuring the angle between two polished surfaces is termed a **goniometer.** The spectrometer represented in Fig. 49 may be used as a goniometer, and by its aid the angles of a prism may be determined.

*First Method.*—Let ABC (Fig. 50) be a principal section of a prism, of which the angle B is required. The prism is placed

on the turn-table of the spectrometer, with its faces AB, BC vertical, while the parallel beam from the collimator falls partly on the face BC, and partly on AB. From each of these faces a parallel beam is reflected, and if either of these beams falls on the object-glass of the telescope, it will be brought to a focus on the cross-wires of the latter. The telescope is adjusted so that an image of the slit is formed on the cross-wires by light reflected from the face BC, and the position of the telescope is read. Without moving the prism, the telescope is rotated till an image of the slit is formed on the cross-wires by light reflected from the face AB, and another reading is taken. The difference between these two readings is equal to twice the angle ABC.

FIG. 50.—Illustrates one Method of determining an Angle of a Prism.

The angle through which the telescope has been rotated is obviously equal to the angle FBG (Fig. 50). Produce DB to E. Then, since the rays DB, BG are equally inclined to BC, it follows that $\angle$ GBC $= \angle$ CBE. Similarly, $\angle$ FBA $= \angle$ ABE. Then—

$$\angle FBG = \angle FBA + \angle ABE + \angle CBE + \angle GBC = 2(\angle ABE + \angle CBE) = 2\angle ABC.$$

*Second Method.*—Adjust the prism and telescope so that an image of the slit is formed on the cross-wires by means of light reflected from one face, BC (Fig. 51), of the prism. Without moving the telescope, rotate the prism until the face AB acquires such a position that light reflected from it forms an image of the slit on the cross-

FIG. 51.—Illustrates a Second Method of determining an Angle of a Prism.

wires. In order to attain this end, it is obvious that the face AB must be rotated until it becomes parallel to the position previously occupied by BC; in other words, if we produce CB to D, the prism must be rotated through an angle ABD, $= \pi - \angle ABC$. Thus, the angle through which the prism is rotated is equal to the supplement of the angle ABC of the prism.

**Adjustment of the Prism.**—Before the foregoing measurements can be made, the prism must be adjusted so that its faces are vertical. This adjustment is effected by the aid of the levelling screws of the central turn-table. Due precaution must be taken that, after one of the faces has been rendered vertical, this adjustment is not disturbed in adjusting the second face. This can be ensured as follows. Let L, K, H (Fig. 50) be the three levelling screws. Adjust the prism by eye so that the edge BC is perpendicular to an imaginary line joining the levelling screws K, H. Then the inclination of the face BC to the horizon can be adjusted by the screws K and H; any subsequent adjustment effected by the screw L will only rotate the table about the horizontal line KH, and will thus merely rotate the face BC in its own plane, without altering its inclination to the horizon. On the other hand, an adjustment of the screw L will serve to render the face AB vertical.

Having placed the prism in the position described, rotate the turn-table till the parallel beam from the collimator is reflected partly from the face BC, and partly from AB, as in Fig. 50. Adjust the screws K and H, till the image of the slit, formed by light reflected from the face BC, is in the middle of the field of the telescope. Then adjust the single screw L, till the image of the slit, formed by light reflected from AB, is in the middle of the field of the telescope. The prism is then completely adjusted.

EXPT. 9.—Adjust the prism supplied to you, in the manner previously described, and measure each of its angles by the two methods given above. Test the accuracy of your result by adding together the three angles of the prism. The sum of the angles should be equal to 180°.

**Determination of the Refractive Index of a Prism.**—When the refracting angle, $\alpha$, has been measured, a determination of the angle of minimum deviation, $\delta$, will enable us (p. 58) to calculate the refractive index, $\mu$, from the equation—

$$\mu = \frac{\sin\{(\alpha + \delta)/2\}}{\sin(\alpha/2)}.$$

Since the value of $\mu$ depends on the colour of the light refracted through the prism, it follows that, for accurate work, light of a particular colour must be used. Monochromatic light (*i.e.* light which corresponds to a very restricted part of the spectrum) may be obtained by introducing common salt into a Bunsen flame. When such a flame (termed a sodium flame) is placed in front of the collimator slit, and the prism and telescope are adjusted to the positions shown in Fig. 48, a single vertical line is seen on looking through the telescope. This is the image of the slit formed by the yellow light emitted by the flame. When the spectrometer is sufficiently powerful, two narrow yellow lines (termed the D lines) are seen ; this shows that the light emitted by the sodium flame is not quite homogeneous, but consists of two constituents differing but slightly in colour. The prism can then be rotated, the telescope being moved so that the cross-wires are kept on the image of the slit. The deviation produced by refraction through the prism is equal to the angular difference between the positions of the telescope when the slit is seen directly, and when it is seen by means of light refracted through the prism. The prism must be rotated until the deviation has its smallest value, when the value of $\delta$ can be obtained. The value of $\mu$ for the yellow light from a sodium flame can then be calculated.

EXPT. 10.—Determine the value of $\mu$, with respect to yellow light, for the glass prism supplied to you.

To determine the refractive index of a liquid, the latter is enclosed in a cell made with plate glass sides. Refraction through the plate glass produces no appreciable effect, so that the procedure is similar to that described above.

**Light of Definite Colour.**—When an electric discharge is passed through a vacuum tube containing a trace of hydrogen gas, light corresponding to a number of restricted portions of the spectrum is emitted. When this light is analysed by means of a spectrometer, the resulting spectrum is seen to consist of a number of narrow coloured lines, each being an image of the slit formed by one of the constituents of the light. The red line is termed the C line, while the greenish-blue line is termed the F line, and the violet line is termed the *h* line. Each one of these lines corresponds to a perfectly definite kind of light, so that we often speak of C light, D light, F light, &c.

When sunlight is analysed by means of a spectrometer, the spectrum, as already stated, consists of a coloured band, shading off from violet at one end, to red at the other. This band is not continuous, but is crossed by a number of narrow black lines, termed **Fraunhofer lines**; each of these corresponds to a particular constituent of pure white light which is missing in sunlight. These lines may also be used to define different positions in the spectrum. Further information on this point will be found in Chap. XIV.

**Dispersive Power.**—Experiment shows that, for an ordinary transparent medium, the refractive index always has a greater value for blue than for red rays, while for rays corresponding to intermediate portions of the spectrum the refractive index has intermediate values. Let $\mu_r$ and $\mu_b$ be the values of the refractive index of a particular medium for red and blue rays respectively. Let $(\mu_r + \mu_b)/2 = \mu$. This value of $\mu$ will correspond to some point in the spectrum intermediate between the red and the blue.

FIG. 52.—Achromatic Combination of Prisms.

Let ABC (Fig. 52) be a principal section of a prism of which the acute angle B is used as the refracting angle, and let $\angle ABC = a$. Then, if $\delta$ is the deviation produced in the rays for which the refractive index is equal to $\mu$, we have (p. 58)—

$$\delta = (\mu - 1)a.$$

Let $\delta_r$ and $\delta_b$ be the deviations corresponding to red and blue lights respectively. Then—

$$\delta_r = (\mu_r - 1)a = \frac{\mu_r - 1}{\mu - 1} \cdot (\mu - 1)a = \frac{\mu_r - 1}{\mu - 1}\delta. \quad . \quad . \quad (1)$$

Similarly—

$$\delta_b = (\mu_b - 1)a = \frac{\mu_b - 1}{\mu - 1}\delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2), we obtain—

$$\delta_b - \delta_r = (\mu_b - \mu_r)a = \frac{\mu_b - \mu_r}{\mu - 1}\delta.$$

The factor $(\mu_b - \mu_r)/(\mu - 1)$ is termed the **dispersive power** of the medium, with respect to the red and blue rays.

**Achromatic Combination of Prisms.**—When white light is transmitted through an ordinary prism, the general direction of

the light is altered, and in addition, the light is decomposed into its constituents, which are deviated to different extents. In other words, deviation and dispersion are both produced. It is possible by using prisms of different substances, to obtain deviation without any great amount of dispersion; a combination of prisms designed for the purpose is said to be **achromatic.**

The dispersive power of flint glass is much greater than that of crown glass. Let the prism ABC (Fig. 52) be supposed to consist of crown glass, of which the refractive indices for red and blue rays are equal to $\mu_r$ and $\mu_b$. Then, if $\angle ABC = \alpha$, the dispersion, $\delta_b - \delta_r$, is given by the equation—

$$\delta_b - \delta_r = (\mu_b - \mu_r)\alpha.$$

Let DAB be a prism of flint glass, of which the refractive indices for red and blue rays are respectively equal to $\mu'_r$ and $\mu'_b$, while $(\mu'_r + \mu'_b)/2 = \mu'$. Then, if $\angle DAB = \alpha'$, while $\delta'_r$, $\delta'_b$, and $\delta'$ are the deviations produced by rays for which the refractive indices are equal to $\mu'_r$, $\mu'_b$, and $\mu'$, we have—

$$\delta'_b - \delta'_r = (\mu'_b - \mu'_r)\alpha'.$$

When white light passes from right to left through the prism ABC, it is deviated downwards, the blue being more deviated than the red light. If the second prism DAB is combined with ABC in the manner represented in Fig. 52, light passing through DAB will be deviated upwards, blue light being more deviated than red light. If the dispersion, $(\delta'_b - \delta'_r)$, produced by DAB, is equal to the dispersion, $(\delta_b - \delta_r)$, of ABC, the resulting spectrum will be folded back on itself so that the red and blue rays are equally deviated. For this to be the case—

$$(\delta'_b - \delta'_r) = (\delta_b - \delta_r)\ ; \ \therefore\ (\mu'_b - \mu'_r)\alpha' = (\mu_b - \mu_r)\alpha.$$

This determines the angles $\alpha'$ and $\alpha$ of the prisms, in order that the transmitted light shall be *approximately* free from colour. The deviation of the transmitted light is equal to—

$$\delta - \delta' = (\mu - 1)\alpha - (\mu' - 1)\alpha'.$$

If the two prisms are made from the same glass, $\mu_b = \mu'_b$, $\mu_r = \mu'_r$, and consequently $\mu = \mu'$. For the dispersions of the two to be equal, $\alpha$ must be equal to $\alpha'$, so that the refracting angles of the prisms are equal. When the two prisms are combined as in Fig. 52, they form a parallel slab of glass, so that both the dispersion and the deviation are annulled.

It is also possible to combine two prisms of different substances, in such a manner that no deviation is produced for the mean rays of the spectrum, while the dispersion is left outstanding. This arrangement is used in constructing direct vision spectroscopes.

**Chromatic Aberration of a Lens.**—Let a lens be made from glass of which the refractive indices, for red and blue rays, are respectively equal to $\mu_r$ and $\mu_b$, while $(\mu_r + \mu_b)/2 = \mu$. Then, if $r_1$ and $r_2$ are the radii of curvature of the surfaces of the lens, the focal length, $f_r$ for red rays, is given by—

$$\frac{1}{f_r} = (\mu_r - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Similarly, the focal length $f_b$, for blue rays, is given by—

$$\frac{1}{f_b} = (\mu_b - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Since $\mu_b$ is greater than $\mu_r$, it follows that $1/f_b$ is greater than $1/f_r$, or $f_b$ is smaller than $f_r$. Thus, **the focal length of a lens is smaller for blue than for red rays, or the two principal foci for blue rays are nearer to the lens than those for red rays.**

A lens acts like a prism, in so far as the incident rays are deviated toward or away from the axis, according as the central or the peripheral portion of the lens is the thicker (p. 69). If the rays are deviated *toward* the axis (convergent lens), then blue rays are deviated more than red rays, and the lens is more strongly convergent for blue than for red rays. If the rays are deviated *away from* the axis (divergent lens), the lens is more strongly divergent for blue than for red rays.

A point source of white light on the axis of a single lens never gives rise to an image at a single point; the image consists of a series of coloured points on the axis, the blue image being nearer to the lens than the red image. Thus, the complete image consists of a small linear spectrum lying along the axis, the blue end of the spectrum being nearer to the lens. This divergence from the laws developed in the last chapter is termed **chromatic aberration.**

If a single lens is used to form an image on a screen, it will, of course, be impossible for the various coloured images to be simultaneously in focus. The blue image, being the nearest to the lens, will be the smallest, so that, if the red rays are focussed on the screen, the red image will have smaller blue, green, &c., images superposed on it,

slightly out of focus. The edge of the resultant image will thus appear red.

Now—

$$\frac{1}{f_r} = \frac{\mu_r - 1}{\mu - 1} \cdot (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\mu_r - 1}{\mu - 1} \cdot \frac{1}{f},$$

where $f$ is the focal length corresponding to rays for which the refractive index of the lens is equal to $\mu$. Similarly—

$$\frac{1}{f_b} = \frac{\mu_b - 1}{\mu - 1} \cdot (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\mu_b - 1}{\mu - 1} \cdot \frac{1}{f}.$$

The chromatic aberration for parallel rays is equal to $f_r - f_b$. Since $f_r$ is greater, and $f_b$ less, than $f$, it follows that, to a first approximation, $f_r f_b = f^2$. Then—

$$\frac{1}{f_b} - \frac{1}{f_r} = \frac{f_r - f_b}{f_r f_b} = \frac{f_r - f_b}{f^2} = \frac{\mu_b - \mu_r}{\mu - 1} \cdot \frac{1}{f};$$

$$\therefore f_r - f_b = \frac{\mu_b - \mu_r}{\mu - 1} \cdot f,$$

or, **the chromatic aberration for parallel rays is equal to the mean focal length of the lens, multiplied by the dispersive power of the substance of which the lens is composed.**

**Achromatic Combinations.**—Let two lenses, of mean focal lengths $f$ and $f$, be placed in contact, and let $\mu_r$, $\mu_b$, and $\mu$ refer to the first lens, while $\mu'_r$, and $\mu'_b$, and $\mu'$ refer to the second lens. Then if $F_r$ is the focal length of the combination for red rays, we have (p. 73)—

$$\frac{1}{F_r} = \frac{\mu_r - 1}{\mu - 1} \cdot \frac{1}{f} + \frac{\mu'_r - 1}{\mu' - 1} \cdot \frac{1}{f'} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

If $F_b$ is the focal length of the combination for blue rays—

$$\frac{1}{F_b} = \frac{\mu_b - 1}{\mu - 1} \cdot \frac{1}{f} + \frac{\mu'_b - 1}{\mu' - 1} \cdot \frac{1}{f'} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

If the focal length of the combination is to have the same value for red and blue rays, $F_r = F_b$. In this case the left-hand sides of (3) and (4) are equal, and subtracting (3) from (4), we obtain—

$$\frac{\mu_b - \mu_r}{\mu - 1} \cdot \frac{1}{f} + \frac{\mu'_b - \mu'_r}{\mu' - 1} \cdot \frac{1}{f'} = 0. \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

When (5) is satisfied, the linear spectrum formed along the axis is folded on itself, its red and blue ends being brought into

coincidence. In practice it is found that the best results are obtained when the yellow D rays, and the bluish-green F rays, are brought to a focus at the same point. In this case the images formed by the brightest rays of the spectrum (the yellow, green, and greenish-blue) are brought into approximate coincidence at the focus. Then $\mu_r$ and $\mu_b$ signify the refractive indices of the first lens for the D and F rays respectively, while $\mu'_r$ and $\mu'_b$ have a similar significance with regard to the second lens.

Since $\mu_b > \mu_r$, and $\mu'_b > \mu'_r$, while $\mu$ and $\mu'$ are both greater than unity, it follows that $f$ and $f'$ must have opposite signs. In the construction of an achromatic telescope objective a convergent lens of crown glass is combined with a divergent lens of flint glass. The crown glass lens is more strongly *convergent* for blue than for red light, while the flint glass lens is more strongly *divergent* for blue than for red light.

Each lens, of course, has two surfaces, so that when the glasses from which the lenses are to be made have been chosen, we have four unknown quantities (the four radii of curvature) to determine. One equation between these four unknown quantities is given by (5). Let F be the required focal length of the combination. Then, (p. 73)—

$$1/F = 1/f + 1/f'. \quad \ldots \ldots \ldots \quad (6)$$

(6) gives a second equation between the four unknowns. In order to avoid loss of light by reflection, it is customary to cement the lenses together with Canada balsam; for this to be possible, the second surface of the first lens must have the same radius of curvature as the first surface of the second lens. This gives us a third equation between the four unknowns. The remaining equation is determined from the condition that the spherical aberration (p. 77) of the lens combination shall be as small as possible. To secure this end, the free surface of the crown glass lens is more strongly curved than the free surface of the flint glass lens, both being convex outwards. The free surface of the crown glass lens faces the incident rays (Fig. 53).

FIG. 53.—Achromatic Combination of Lenses.

According to Herschel, the best form of a telescope objective, of mean focal length F, is obtained by making the radii of curvature of the

free surfaces of the crown and flint glass lenses, respectively equal to $0{\cdot}672 \times F$ and $1{\cdot}420 \times F$, the radii of the remaining surfaces being calculated from (5), p. 95, and (6), p. 96. The free surface of the flint glass lens is often made plane.

The following table gives the refractive indices of a number of samples of different kinds of crown and flint glass, made by Messrs. Chance, for the D and F rays :—

| | D | F | | D | F |
|---|---|---|---|---|---|
| Soft Crown . . | 1·5146 | 1·5210 | Dense Flint . | 1·6224 | 1·6347 |
| Hard Crown . | 1·5171 | 1·5231 | Extra Dense Flint . . . | 1·6504 | 1·6642 |
| Extra Light Flint. . . | 1·5410 | 1·5491 | Double Extra Dense Flint | 1·7102 | 1·7273 |
| Light Flint . . | 1·5740 | 1·5839 | | | |

PROBLEM.—An achromatic objective, of focal length = 30 cms., is to comprise two thin lenses, cemented together with canada balsam, and made from Chance's "Hard Crown" and "Dense Flint" glass respectively. The free surface of the divergent lens is to be plane. Calculate the radii of curvature of both lenses.

Let $f$ = focal length of convergent lens of "Hard Crown" glass
$f'$ = ,, ,, divergent lens of "Dense Flint" glass
Then $1/f + 1/f' = -1/30$ . . . . . . . (1)

For **Hard Crown** glass :—
Mean refractive index, $\mu = (1{\cdot}5171 + 1{\cdot}5231)/2 = 1{\cdot}5201$.
Dispersive power $= (1{\cdot}5231 - 1{\cdot}5171)/0{\cdot}5201 = 0{\cdot}0115$.

For **Dense Flint** glass :—
Mean refractive index, $\mu' = (1{\cdot}6224 + 1{\cdot}6347)/2 = 1{\cdot}6285$.
Dispersive power $= (1{\cdot}6347 - 1{\cdot}6224)/0{\cdot}6285 = 0{\cdot}0196$.

Then from (5), p. 95—
$0{\cdot}0115/f + 0{\cdot}0196/f' = 0$ . . . . . . . (2)

$$\therefore \frac{1}{f} = -\frac{196}{115}\frac{1}{f'},$$

and from (1) above

$$\left(-\frac{196}{115} + 1\right)\frac{1}{f'} = -\frac{1}{30}. \quad \therefore f' = 21{\cdot}13 \text{ cms.}$$

$$\text{also, } f = -\frac{115}{196} \times 21{\cdot}13 = -12{\cdot}39 \text{ cms.}$$

Let the concave surface of the divergent lens have a radius of curvature denoted by R; then, since the other surface of this lens is plane, we have—

$$1/f' = (\mu' - 1)/R \quad \therefore R = (\mu' - 1)f' = 13{\cdot}28 \text{ cms.}$$

Since the lenses are cemented together, one surface of the convergent lens must also have a radius of curvature R = 13·28 cms.; let the other (free) surface of this lens have a radius of curvature denoted by $r$.

Then
$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{R}\right),$$

and
$$\frac{1}{r} = \frac{1}{(\mu - 1)f} + \frac{1}{R} = -\frac{1}{0{\cdot}5201 \times 12{\cdot}39} + \frac{1}{13{\cdot}28} = -0{\cdot}0799.$$

$$\therefore r = -12{\cdot}52 \text{ cms.}$$

**Achromatic Microscope Objective.**—A microscope objective of high power generally consists of a number of lenses, made from different kinds of flint and crown glass, or sometimes of Jena glass. An objective designed by Prof. Abbe, and made by Zeiss, is represented in Fig. 54. The function of the lowest lens, which is hemispherical, has already been described (p. 78). Of the remaining lenses, those which are divergent are made from different kinds of flint glass, while the convergent lenses are made from different kinds of crown glass. Each lens after the first one, produces an extra fold in the spectrum.

FIG. 54.—Achromatic Microscopic Objective.

**Method of testing a Telescope Objective.**—Foucault invented the following very accurate method of testing a lens for chromatic and spherical aberration. Light from a star, or other distant source, is brought to a focus, F (Fig. 55), by the lens to be tested. A screen, S, is placed so that its edge just covers the image formed at F. A telescope, T, is placed just behind the screen, and focussed on the surface of the lens L. If all of the refracted rays converge to a single point at F, the lens L will appear quite dark. If, on the other hand, some rays are brought to a focus in front of, or behind, F, a certain number of these rays will reach the telescope and render the surface of the lens L luminous. If chromatic

aberration exists, the lens will appear coloured, the colour changing if the screen S is slightly displaced. If spherical aberration exists, the peripheral portions of L will appear bright when the central portion is dark, or *vice versâ*.

FIG. 55.—Foucault's Method of testing a Lens

The above method has been extended by Toepler, for the purpose of detecting small local differences in the refractive index of a medium. A luminous gas flame, partly hidden by a screen with a vertical straight edge, is used as a source of light, and an image of this is formed by a perfect achromatic lens. A second screen with a vertical straight edge is adjusted so as just to cover the image, the second straight edge just coinciding with the image of the first straight edge. A telescope is placed behind the screen as in Fig. 55, and focussed on a point near the surface of the lens. When the medium between the lens and screen is homogeneous, the lens appears dark; but any local variation in the medium will cause some of the rays to go astray and escape the second straight edge. Using an electric spark instead of a gas flame, Prof. Wood has been able to obtain instantaneous photographs of sound-waves in air by this method.

**Two Lenses at a Definite Distance apart.**—We have already (p. 76) found an expression for the focal length of a single lens (the equivalent lens) which will produce an image of the same size as that formed by a combination of two lenses separated by a definite distance $d$. Let us now suppose that the lenses forming the combination are composed of the same substance, of which the refractive indices, for red and blue rays, are equal to $\mu_r$ and $\mu_b$, while $(\mu_r + \mu_b)/2 = \mu$. Let $F_r$ and $F_b$ be the focal lengths of the equivalent lenses for red and blue rays. Then, if $f_1$ and $f_2$ are the focal lengths of the lenses forming the combination, for rays corresponding to the refractive index $\mu$, we have, from p. 76, together with the results arrived at on p. 95—

$$\frac{1}{F_r} = \frac{\mu_r - 1}{\mu - 1}\left\{\frac{1}{f_1} + \frac{1}{f_2}\right\} + \left(\frac{\mu_r - 1}{\mu - 1}\right)^2 \frac{d}{f_1 f_2} \quad . \quad . \quad (7)$$

Similarly—

$$\frac{1}{F_b} = \frac{\mu_b - 1}{\mu - 1}\left\{\frac{1}{f_1} + \frac{1}{f_2}\right\} + \left(\frac{\mu_b - 1}{\mu - 1}\right)^2 \frac{d}{f_1 f_2}. \quad . \quad . \quad . \qquad (8)$$

Let us now find the condition that the focal lengths, $F_r$ and $F_b$, of the equivalent lenses shall be equal. In this case the left-hand sides of (7) and (8) are equal. Subtracting (7) from (8), we obtain—

$$\frac{\mu_b - \mu_r}{\mu - 1}\left\{\frac{1}{f_1} + \frac{1}{f_2}\right\} + \frac{(\mu_b - 1)^2 - (\mu_r - 1)^2}{(\mu - 1)^2} \cdot \frac{d}{f_1 f_2} = 0.$$

Then, since—

$$\{(\mu_b - 1)^2 - (\mu_r - 1)^2\} = \{(\mu_b - 1) - (\mu_r - 1)\}\{(\mu_b - 1) + (\mu_r - 1)\}$$
$$= (\mu_b - \mu_r)(\mu_b + \mu_r - 2) = 2(\mu_b - \mu_r)(\mu - 1),$$

we have—

$$\frac{\mu_b - \mu_r}{\mu - 1}\left\{\frac{1}{f_1} + \frac{1}{f_2} + \frac{2d}{f_1 f_2}\right\} = 0.$$

Discarding the common factor $(\mu_b - \mu_r)/(\mu - 1)$, and simplifying, we obtain—

$$d = -(f_1 + f_2)/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

**Equation** (9) **gives the condition that the combination of two lenses, of mean focal lengths** $f_1$ **and** $f_2$, **separated by a distance** $d$, **shall be equivalent** (p. 74), **for red and blue rays, to single lenses with equal focal lengths.**

Since $d$ is an essentially positive quantity, it follows that $(f_1 + f_2)$ must be negative, so that one or both of the combined lenses must be convergent.

It is often found stated in text-books on Light that when (9) is satisfied, the lens combination is achromatic, in the sense that the red and blue images which it forms are equal in size, though they are not formed in the same position. A little consideration will show that this statement, in its general form, is erroneous. For let F be the common value of $F_r$ and $F_b$. Then, for red rays, the equivalent lens, of focal length F, must be placed at a distance

$$Fd\frac{\mu_r - 1}{\mu - 1} \cdot \frac{1}{f_2}$$

behind the first lens of the combination (p. 76), while for blue rays the equivalent lens must be placed at a distance

$$Fd\frac{\mu_b - 1}{\mu - 1} \cdot \frac{1}{f_2}$$

behind the first lens of the combination. Thus, the equivalent lenses have different positions, and if the object is achromatic, it will be at different distances from them. From equation (9), p. 72, the magnification produced by a lens depends, not alone on the focal length, but also on the distance of the object from the lens; thus it follows that the blue and red images will, in general, differ in size. When the object is at a great distance from the lens combination, equation (9), p. 72, shows that the images will be equal in size, since the values of $u$ will then scarcely be affected by the small distance between the positions of the equivalent lenses; but a lens combination of the kind described is seldom used to form an image of a distant object.

Two simple lenses, separated by a definite distance, are generally used in the construction of telescope and microscope eye-pieces. In such cases we are concerned, less with the absolute magnitudes and positions of the coloured images, than with the angles which these images subtend at the eye. Accordingly, eye-pieces are constructed so that the various coloured images subtend equal angles at the eye. We shall return to this point in Chap. X.

## RAINBOWS

**Characteristics of Rainbows.**—Rainbows are seen when the sun shines, for example, on falling rain, or on the spray from a cascade or wave. For rainbows to be seen, the observer's back must, in all cases, be turned toward the sun; hence, from very early times, the formation of rainbows has been attributed to the refraction and internal reflection of sunlight by small drops of water. In favourable circumstances, several bows may be seen. The brightest bow is termed the **primary bow**; its radius subtends an angle of about 41° at the observer's eye, and it exhibits the brilliant colours of the solar spectrum, being red on its outer, and violet on its inner edge. A larger and fainter bow, of which the radius subtends an angle of about 52° at the observer's eye, is often seen. This bow is red on its inner, and violet on its outer, edge, and is termed the **secondary bow**. Other faint bows, termed **supernumerary bows**, are sometimes seen just within the primary bow. A general account of the formation of the primary and secondary bows will now be given; for an account of the formation of the supernumerary bows, more advanced treatises, such as Preston's *Theory of Light*, may be consulted.

The primary bow is formed by rays each of which has been refracted into, and out of, a drop of water, having meanwhile suffered *one* internal reflection. The secondary bow is formed by rays each of which has been *twice* internally reflected between its entrance into and emergence from a drop of water.

**Deviation of a Ray once internally reflected in a Transparent Sphere.**—Let SA (Fig. 56) be a ray incident at an angle $i$ on a transparent sphere, such as a drop of water. Draw the radius OA of the sphere. Then the incident ray makes an angle $i$ with OA produced. The ray AB, refracted into the

FIG. 56.—Ray entering a Transparent Sphere, and emerging after One Internal Reflection.

sphere, makes an angle $OAB = r$ with the radius OA, in accordance with the equation—

$$\sin r = \sin i/\mu.$$

Let the ray AB be incident at B on the back surface of the sphere. Draw the radius OB. Then the angle OBA is the angle of incidence at B. Further, since $OB = OA$, the triangle OAB is isosceles, and $\angle OBA = \angle OAB = r$. Thus, the ray AB is incident at an angle $r$ at B, and, if BC is the corresponding reflected ray, $\angle CBO = \angle OBA = r$.

Let the ray BC be incident at C on the front surface of the sphere. Draw the radius OC. Then, since $OB = OC$, the angle of incidence OCB is equal to $\angle OBC$, or to $r$. Consequently, the emergent ray CE is inclined to the radius OC (produced) at an angle $i$, equal to the angle of incidence of the ray SA at A.

Produce the rays SA and EC to meet at D. Then the

deviations produced by refraction at A, reflection at B, and refraction at C, are together equal to the angle D. This is the angle through which the incident ray SD must be rotated about the point D, in order to bring it into coincidence with the direction of the emergent ray DE.

It is easy to find an expression for the deviation D. It will be noticed that at each of the points A, B, and C, the ray is deviated in the same sense, so that the deviations at A, B, and C are additive. Let us term a rotation in the sense in which the hands of a clock revolve a **right-handed rotation**. Then, at A the ray is deviated through an angle $(i-r)$, in a right-handed direction. The angle ABC is equal to $2r$, so that, in order to bring the ray AB into the direction BC, by a right-handed rotation about B, it must be rotated through an angle of $(180° - 2r)$. To bring the ray BC into the direction of the emergent ray CE, it must be rotated, in a right-handed direction, through an angle $(i-r)$. Thus, the resultant deviation D, which is equal to the sum of the deviations at A, B, and C, is given by—

$$D = (i - r) + (180° - 2r) + (i - r) = 180° + 2i - 4r.$$

**Angle of Minimum Deviation.**—If parallel rays are incident on a sphere, the ray directed toward the centre O will be incident normally, so that for this ray $i = 0$. Rays incident tangentially on the sphere will correspond with $i = 90°$. Thus, rays will be incident on the sphere at all angles between 0° and 90°. For any particular angle of incidence, $i$, we can calculate the value of the angle of refraction, $r$, and substituting the corresponding values of $i$ and $r$ in the equation—

$$D = 180° + 2i - 4r, \quad . \ . \ (a)$$

we can determine the final deviation, D. In Fig. 57, a curve is given showing the values of D for values of $i$ between 0° and 90°. It will be seen that for an angle of incidence

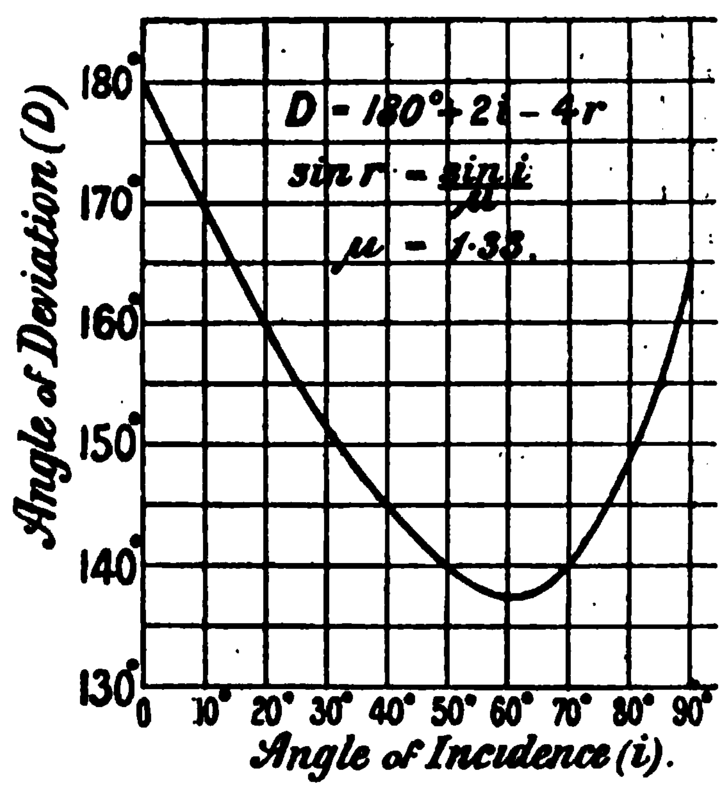



FIG. 57.—Deviations corresponding to Various Angles of Incidence.

equal to about 61°, the resultant deviation has its minimum value, equal to about 138°, or (180° − 42°). The ray incident normally on the sphere, ($i = 0$), has its direction reversed by the internal reflection, so that its deviation amounts to 180°. The ray incident tangentially ($i = 90°$), suffers a deviation of about 164°.

Since no ray is deviated by less than (180° − 42°), it follows that the rays emerging from the sphere are all contained within a right circular cone, half the vertical angle of which is equal to 42°. It will also be noticed from Fig. 57, that the deviation changes comparatively slowly in the neighbourhood of the lowest point on the curve; consequently, the emergent rays will be more closelv packed in the neighbourhood of the surface of the cone, than anywhere else within it. Thus, **the emergent rays will be contained within a cone, half the vertical angle of which is equal to 42°, the greater proportion of them lying near to the surface of this cone** (Fig. 58).

FIG. 58.—Rays emerging after One Internal Reflection.

The curve in Fig. 57 has been constructed for a value of $\mu$ equal to 1·33, the mean refractive index of water. But for water, the value of $\mu$ is greater for violet than for red rays. Thus, if a ray of white light is incident at an angle $i$ on a transparent sphere, the value of $r$ will be less for the violet than for the red rays, and consequently, from equation ($a$) above, the resultant deviation will be greater for the violet than for the red rays. In other words, the violet rays will be contained within a cone of smaller vertical angle than that containing the red rays. Thus, if a white screen is placed in front of the sphere, a circular coloured band will be formed by the densely packed minimum deviation rays, the outside of this band being red, and the inside violet. The circular space enclosed by the band will be slightly illuminated.

EXPT. 11.—Place a spherical flask containing water in front of an illuminated aperture in a white screen. A brilliant circular band, blue inside and red outside, will be formed on the screen. This band is formed by the least deviated rays, which have suffered two refractions and one internal reflection in the flask.

**Deviation of Rays twice internally reflected in a Transparent Sphere.**—The ray BC (Fig. 56), incident internally at the point C of the sphere, will give rise to a reflected ray, inclined to the radius OC at an angle $r$. This reflected ray (not shown in Fig. 56) will again fall on the surface at an angle $r$, and give rise to a refracted ray inclined to the radius at an angle $i$. The deviation produced on entering, and on leaving, the sphere is equal to $(i - r)$, as before. At each reflection the deviation is equal to $(180 - 2r)$. Thus, the total deviation, D, is given by the equation—

$$D = 2(i - r) + 2(180^\circ - 2r) = 360^\circ + 2i - 6r.$$

A curve can be drawn showing the connection between D and $i$. When this is done, it is found that the form of the curve is similar to that given in Fig. 57. For a certain value of $i$, the deviation D has a minimum value, equal to 232°, or 360° − 128°. The ray directed toward the centre of the sphere suffers two internal reflections at normal incidence, so that its direction is twice reversed, and the deviation amounts to 360°. The remaining rays are contained in the space exterior to a cone, half the vertical angle of which is equal to 180° − 128° = 52° (Fig. 59). The rays are more closely packed in the neighbourhood of the surface of this cone than elsewhere. Further, the vertical angle of the cone for red rays is smaller than that for violet rays (Fig. 59). Thus, if white light, after being twice internally reflected within a sphere, is allowed to fall on a white screen, it will paint a circular band subtending an angle of 2 × 52° = 104° at a point behind the sphere, the outside of the band being violet, and the inside red.

FIG. 59.—Rays emerging after Two Internal Reflections.

**Formation of Rainbows.**—When sunlight shines on falling rain, cones of rays similar to Figs. 58 and 59 will leave each drop of water. Only a limited number of rays in the cone derived from a particular raindrop can reach the eye of an observer;

the rainbow seen is formed by rays from a great number of drops, possessing certain positions with respect to the eye of the observer.

Let E (Fig. 60) be the position of the eye of the observer, and let a number of raindrops be situated at O, $O_1$, $O_2$, $O_3$, in a vertical line. Let us first consider the effect produced by the light which has suffered only one internal reflection. Let EC be drawn parallel to the rays from the sun. Draw $EO_1$, making an angle of 42° with EC. Then the raindrop at $O_1$ will send some of its least deviated rays to E, and the point $O_1$ will appear bright. A raindrop at O, a point below $O_1$, will send only the more deviated, and less densely packed, rays to E, so that points below $O_1$ will be only faintly illuminated. Points above $O_1$ will send to E no light which has suffered only one internal reflection.

Fig. 60.—Illustrates the Formation of the Primary and Secondary Bows.

If we now suppose the line $EO_1$ to rotate about EC, the point $O_1$ will describe a circle, such that all raindrops situated in it will send light to E. This circle will, therefore, appear bright. Since the supplement of the angle of minimum deviation for red rays is about 43°, while that for violet rays is about 41°, it is obvious that the bow actually seen will be coloured, the red edge of the bow, which is outside, subtending an angle of $2 \times 43° = 86°$ at the eye of the observer, while the internal violet edge of the bow subtends an angle of $2 \times 41° = 82°$ at the eye of the observer. This accounts for the primary rainbow.

Let us now consider the light which has been twice internally reflected in a raindrop. From E draw $EO_2$, making an angle of 52° with EC. Then, a raindrop at $O_2$ will send to E some of the least deviated of the rays which have been twice internally reflected. Thus, the point $O_2$ will appear bright. Points above $O_2$ will send only the more deviated, and less densely packed, rays to E, so that points above $O_2$ will be only faintly illuminated. Points between $O_2$ and $O_1$ will send no rays to E. If we suppose the line $EO_2$ to rotate about EC, the point $O_2$ will

describe a circle such that all raindrops on it will send bright light to E. Since the least deviated red rays make an angle of 51°, while the least deviated violet rays make an angle of 54°, with the incident rays, it follows that the bow actually seen will be coloured, its inner edge, which is red, subtending an angle of $2 \times 51° = 102°$ at the eye of the observer, while its outer violet edge subtends an angle of $2 \times 54° = 108°$ at the eye of the observer. This accounts for the secondary rainbow. The space between the primary and secondary rainbows appears darker than the rest of the sky, as is indicated by theory.

Owing to the finite apparent magnitude of the sun, all rays incident on the raindrops are not parallel. As a consequence, overlapping occurs, and the colours of the rainbow are not pure. In hazy weather, when the apparent magnitude of the sun is greatly increased by the scattering of light by the mist, *white* rainbows are sometimes formed. This is merely an extreme effect of the overlapping of different colours.

## Questions on Chapter IV

1. A ray of homogeneous light is incident at an angle $\phi$ on a prism of angle $i$, and the deviation D is observed. Prove that $\chi$, the angle of emergence, may be found from the formula $\chi = D + i - \phi$.

2. Given a prism of a substance of known index of refraction, show how to calculate the deviation produced by it under any given circumstances, especially when the ray goes through the prism symmetrically. Given that the angle of a prism is 60°, and that the minimum deviation it produces with sodium light is 30°, what is the index of refraction of its substance for this kind of light?

3. A ray of light is refracted through a prism in a plane perpendicular to its edge. Prove that if the angle of incidence is constant, the deviation increases with the angle of the prism. What is the limiting angle of the prism, such that the incident ray does not emerge when it meets the second face of the prism?

4. Give a detailed account of the method of finding by experiment the refractive index of the material of a transparent solid or liquid prism.

5. Explain how two prisms of different refractive indices and dispersive powers may be combined to form an achromatic combination.

6. Explain the theory and construction (1) of an achromatic object-glass; (2) of an achromatic eye-piece formed of two convex lenses.

7. Give a general explanation of the construction of an achromatic lens, with a diagram to show the paths of rays incident parallel to the axis through each of the component lenses when used separately.

8. A thin convex lens of crown glass and a thin concave lens of flint glass form an achromatic combination when placed in contact. A beam of white light, which is in each case parallel to the axis, falls in different experiments (1) upon the convex lens alone, (2) upon the concave lens alone, (3) upon both lenses in contact. Draw diagrams indicating the paths of the blue and red constituents of the white light in each case.

9. What is meant by an achromatic combination of lenses? You are given a convex lens, and a prism of the same specimen of crown glass, also a prism of flint glass. What observations would you make in order to determine the focal length of a lens of the flint glass which will form, with the crown glass lens, an achromatic object-glass?

10. Two thin lenses are in contact, and form an achromatic combination, one being equi-convex, and the other equi-concave. Calculate from the following data the radius of curvature of each surface of the concave lens, and the focal length of the combination :—

Convex lens.—Radius of curvature, 10 cms.; refractive indices: for red, 1·480; for violet, 1·499.

Concave lens.—Refractive indices: for red, 1·610; for violet, 1·667.

11. A convex lens of focal length 40 cms. is placed in contact with a concave lens of focal length 66 cms. Trace the path of a pencil of rays through the combination from an object at a distance of 200 cms., and state for what purpose such a combination is used.

12. Calculate the focal lengths of the components of an achromatic lens to be of 2 metres focal length, from experiments on prisms of given angles made of the kinds of glass to be used in making the components. Choose the points of agreement in the spectrum for your calculation, and give reasons for your choice.

13. If the refractive indices for red and blue light respectively be 1·525 and 1·542 in a crown glass, and 1·628 and 1·660 in a flint glass, calculate the mean focal length of a flint glass lens which will correct the chromatic aberration of a convex crown glass lens of 50 cms. mean focal length. What will be the focal length of the combination when the two lenses are placed close together?

14. Explain, with the aid of carefully drawn figures, how the primary and secondary rainbows are formed.

15. Find the dispersion produced by a thin prism of angle 15°, having a refractive index for red light of 1·5, and for violet light of 1·6.

### Practical

1. Measure the angle between the surfaces of a given prism by means of a spectrometer.

2. Find the refractive indices, for the three (bright) hydrogen lines, of the given prism, being given a spectrometer, induction coil, vacuum tube, &c.

3. Measure the minimum deviation produced by the given prism.

4. Find, by observations on the two given prisms, their respective dispersive powers.

# CHAPTER V

## OPTICAL CONSTANTS OF MIRRORS AND LENSES

**Introductory**.—The properties of a spherical mirror are completely determined when its radius of curvature, or its focal length, is known. These two magnitudes are not independent; the focal length is equal to half the radius of curvature, both as to magnitude and sign (p. 33).

A lens possesses four optical constants, so related that when any three are known, the fourth can be determined by calculation. These constants are:—The radii of curvature of the two surfaces of the lens, the focal length of the lens, and the refractive index of the substance of which the lens is composed.

If $f$ is the focal length (second focal distance) of a lens, of which the radii of curvature are equal to $r_1$ and $r_2$, while $\mu$ is the refractive index of the substance of which the lens is composed, then (p. 69)—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots\ldots \quad (1)$$

In using this equation, the student should exercise great care in attaching the proper signs to the quantities $f$, $r_1$, and $r_2$.

Each surface of a lens may be treated as a spherical mirror, so that the same methods may be used for determining the radii of curvature of mirrors and lenses. The method of determining the focal length of a lens depends on the character of the lens. The refractive index, $\mu$, of a lens may be determined by substituting the values of $f$, $r_1$, and $r_2$, in (1).

**The Optical Bench**.—The experimental determinations described in the present chapter may often be facilitated by mounting the mirror, or

lens, on a stand, which can be moved backwards or forwards in a straight line. An arrangement to secure rectilinear motion of the stand is termed an **optical bench.** This may consist of a board from one to two yards long, and about six inches wide, with boxwood metre scales screwed down end to end on its upper surface, near one edge. Several stands, consisting of uprights fixed in square wooden blocks, and provided with attachments for lenses and screens, are required; the boxwood scales at the edge of the optical bench serve to guide the wooden blocks.

Distances may be measured by means of a separate boxwood metre scale. Short distances may be measured by the aid of a pair of dividers.

As a source of light, an Argand or Welsbach gas burner may be used, the glass chimney being surrounded by a cylinder of thin sheet zinc, with a circular aperture half an inch in diameter cut in it. A cardboard screen, with a small circular aperture cut in it, may be placed in front of the source of light. Two fine wires, or fine glass fibres, may be stretched across the aperture in the cardboard screen.

## Focal Length of a Lens

**Simple Illustrative Experiment.**—The following experiment brings the properties of divergent and convergent lenses prominently into view.

Expt. 12.—Draw a line, AB (Fig. 61), on a sheet of millboard, and cut a slot perpendicular to this line so that a lens, L, can stand upright in it, half above and half below the surface of the millboard. The line AB should coincide with the axis of the lens. Stick two pins upright in the millboard, on one side of the lens, at points C and D, equidistant from, and on the same side of, the axis AB. Look through the lens, and adjust the position of the eye so that one pin is seen in front of the other. Place pins at E and G, so that all four pins are seen in the same straight line, and draw a straight line through G and E. If this line cuts the axis at F, the distance from L to F gives the

Fig. 61.—Illustrates Experiment 12.

focal length of the lens. It is best to use a lens of from 30 to 70 cms. focal length, and an aperture of from 5 to 7 cms., for this experiment. If the lens is convergent, the line EG will cut the axis on the side of the lens remote from the pins D, C. In this case, for the pins D and C to be seen through the lens, they must be placed at a distance from the latter less than the focal distance.

### Direct Determination of Focal Length (Convergent Lens).—

EXPT. 13.—By means of the convergent lens, form an image of a remote object (such as a distant stack of chimneys), on a sheet of millboard. When the image is most distinct, the surface of the millboard is in the second focal plane of the lens; if the lens is thin, the distance from the millboard to the lens gives the focal length of the latter.

Even when another method of determining the focal length of a convergent lens is used, the results obtained should be checked by the above method.

EXPT. 14.—Place the lens to be examined on the surface of a piece of good looking-glass laid on a table. Above the lens, support a pointed piece of white paper, and observe the real inverted image of this formed by rays refracted through the lens, reflected from the looking-glass, and then again refracted through the lens. Adjust the position of the piece of paper till its point, and the point of the image, coincide. When this adjustment is completed, there will be no parallax between the paper and its image. Then the strip of paper is at the first principal focus of the lens.

The formation of the final image may be thus explained. Since the piece of paper is at the first principal focus of the lens, the first refraction forms a virtual, erect image at an infinite distance *above* either the lens or the mirror. By reflection, a virtual, erect image is formed at an infinite distance *below* the mirror; this image gives rise, by the second refraction, to a real inverted image at the principal focus of the lens.

If the lens is thin, the distance from it to the piece of paper gives the focal length of the lens. If the lens is thick, or if a compound lens (such as a photographic objective) is used, the *position* of the first principal focus with respect to the nearer surface of the lens, or lens combination, can be found in this manner.

This method can be advantageously used in connection with long focus spectacle lenses.

**Observation of Conjugate Foci (Convergent Lens).—**

EXPT. 15.—Place the lens at such a distance from an illuminated aperture that a real image is formed, and determine the position of the image. This can be accomplished, if the experiment is performed in a dark room, by adjusting a white screen so that the image is formed on it. Greater accuracy can be secured by observing the image directly, and adjusting a needle so that there is no parallax between it and the image; this method can be used in an undarkened room.

Let the numerical value of the distance from the aperture to the lens be equal to $d_1$, while that of the distance from the lens to the image is equal to $d_2$. Then $u = d_1$, while $v = -d_2$. Then, if $f$ is the focal length of the lens—

$$\frac{1}{f} = -\frac{1}{d_2} - \frac{1}{d_1}.$$

This method only applies to *thin* convergent lenses.

**Thick Lenses, and Combinations of Lenses.**—It will be proved in a subsequent chapter that, on the axis of a thick lens, or a combination of two or more coaxial lenses, there are two fixed points, termed the **first** and **second principal points**, which possess the following properties: If $u$ is the distance of the object, *measured from the first principal point*, while $v$ is the distance of the corresponding image, *measured from the second principal point*, and $f$ is the distance of the second principal focus, *measured from the second principal point*; then—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \ldots\ldots\ldots\ldots \quad (2)$$

In the case of a thin lens, the two principal points are practically coincident with the point which we have termed the pole of the lens (p. 67).

It thus becomes obvious that a single formula can be used for a lens, whether thin or thick, or for a combination of lenses.

When the lens is convergent, and a real image of an illuminated aperture is formed on a screen, let $u = d_1$, while $v = -d_2$. Then—

$$-\frac{1}{d_2} - \frac{1}{d_1} = \frac{1}{f}.$$

Let us now suppose that the distance between the illuminated aperture, and the screen on which the image is formed, is kept

constant. This distance is equal to $(d_1+d_2)$, *plus* the distance between the two principal points of the lens. If $d_2$ and $d_1$ are unequal, there will always be two positions, in either of which the lens can be placed so as to form an image on the screen. For let $u = d_2$, while $v = -d_1$. These values will obviously satisfy (2). In one of these positions, the lens is nearer to the aperture than to the screen, while in the other position, the lens is nearer to the screen than to the aperture. On diminishing the distance between the aperture and screen, it will be found that the two positions in which the lens can be placed, so as to form an image on the screen, are closer together than previously; and by continuously diminishing the distance between the aperture and the screen, it will at last be found that there is only one position in which the lens can be placed so as to form an image on the screen. In this case $d_2 = d_1 = d$ (say), and the object is as far from the first principal point of the lens as the image is from the second principal point. Then—

$$-\frac{1}{d}-\frac{1}{d}=-\frac{2}{d}=\frac{1}{f}; \therefore d=-2f.$$

If the lens is thin, the distance from the aperture to the screen is numerically equal to $2d$ or $4f$.

In the case of a thick lens, or combination of lenses, we know that the second principal focus is at a distance $f$ from the second principal point; while the screen, when adjusted as described above, is at a distance $2f$ from the second principal point. Thus the screen is at a distance $f$ from the second principal focus. Having determined, by the method described in Expt. 14, the position of the second principal focus relative to the nearer surface of the lens, or combination of lenses, the distance from this point to the screen can be measured; this gives us the **focal length**, $f$, of the lens. If we measure a distance equal to $f$, from the second principal focus, toward the lens, then we determine the position of the **second principal point** of the lens. If the position of the principal focus on the opposite side of the lens has been determined, a measurement of the distance $f$ from this point toward the lens will determine the **first principal point** of the lens.

EXPT. 16.—Determine the focal length of a thin convergent lens by the method just described.

EXPT. 17.—Determine the focal length, and the principal points, of the focussing lens of a magic lantern, or a photographic objective, by the method just described.

**Magnification Method (Convergent Lens).**—

EXPT. 18.—Place two scales, photographed on glass from the same negative, on opposite sides of the lens to be tested. The photographic films should be uncovered, and turned toward the lens. Adjust the scales so that a real image of one is formed on the other; this adjustment should be tested by an observation through a magnifying glass, when parallax between one scale and the image of the other can be easily detected. By observing points of coincidence between the one scale and the image of the other, determine the magnification $m_1 = \mathrm{i}/\mathrm{o}$. Then, if $v_1$ is the distance of the image from the second principal point of the lens we have (p. 72)—

$$m_1 = \frac{\mathrm{i}}{\mathrm{o}} = \frac{f - v_1}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the lens is thin, $v_1$ can be measured, when—

$$f = v_1/(1 - m_1).$$

If the lens is thick, we cannot measure $v_1$ until we know the second principal point of the lens. In this case, having obtained the magnification $m_1$, corresponding to a particular position of the image (which must be noted), shift the scales into new positions, conjugate as before, but corresponding to a magnification $m_2$. Then, if $v_2$ is the (unknown) distance of the image from the second principal point—

$$m_2 = \frac{f - v_2}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtracting (4) from (3), we obtain—

$$m_1 - m_2 = \frac{v_2 - v_1}{f}.$$

Here $m_1$ and $m_2$ are known, and $(v_2 - v_1)$, which is the distance between the images formed in the two experiments, can be directly measured. Thus, $f$ is determined, and the principal points can be found in the manner described in connection with Expt. 17.

**Focal Length of Divergent Lens.**—Thick divergent lenses, or divergent lens combinations, are seldom used. Consequently we shall here confine our attention to *thin* divergent lenses.

EXPT. 19.—Place the divergent lens in contact with a convergent lens of such short focus that the combination is convergent. Determine the focal length, F, of the combination, by one of the methods

used in Expts. 13, 14, or 15. Then, (p. 73), if $f_1$ is the known focal length of the convergent lens, the focal length $f$ of the divergent lens is determined from the equation—

$$1/f = 1/F - 1/f_1.$$

(Remember that F and $f_1$ are both negative.)

EXPT. 20.—Mount the divergent lens so that a distant object can be seen through it. The image seen will be at the principal focus of the lens. The position of this image can be determined in a manner similar to that used in Expt. 5 (p. 54). Place a thin sheet of plate-glass between the lens and the eye, and adjust a small pointed gas flame, so that its image, reflected in the plate-glass, coincides in position with the image of the distant object seen through the lens. Then the principal focus of the lens is as far behind the plate-glass, as the pointed gas flame is in front of it.

## OPTICAL METHODS OF MEASURING CURVATURE

**Concave Mirror, or Concave Surface of Lens.—**

EXPT. 21.—Turn the concave surface toward a distant object, and, by tilting it slightly, form a reflected image of the object on a piece of white card. When the image is most definite, the distance from the surface to the card is equal to the focal length of the surface, or half the radius of curvature.

Rays diverging from the centre of curvature of a concave mirror fall on the mirror normally, so that their directions are reversed, and the reflected rays converge toward the centre of curvature. A small luminous object, placed at the centre of curvature of a concave mirror, gives rise to a real inverted image, also situated at the centre of curvature.

EXPT. 22.—Turn the concave surface to be tested toward an illuminated aperture in a white screen. Adjust so that a distinct image of the aperture is formed on the screen, near the aperture. Then the distance from the surface to the aperture is equal to the radius of curvature.

EXPT. 23.—Place a pointed piece of white paper at right angles to the axis of the surface to be tested, and observe the real inverted image formed. Adjust so that the point of the paper and the point of the image coincide; examine both through a magnifying glass, to see if any parallax can be detected. When the above adjustment has been completed, the distance from the pole of the surface to the point of the paper is equal to the radius of curvature of the surface.

**Convex Mirror, or Convex Surface of Lens.**—If a pencil of rays converges toward the centre of curvature of a convex reflecting surface, each ray will fall normally on the surface, and its direction will be reversed by reflection.

EXPT. 24.—Place a convergent lens in front of an illuminated aperture in a white screen, in such a position that a real image of the aperture is formed. Adjust a needle so that its point coincides with the real image of the aperture. Then place the convex surface to be tested between the needle and the lens, and adjust until an image of the aperture is formed on the screen, beside the aperture. In this case the rays fall normally on the convex surface, and the distance between the surface and the needle-point is equal to the radius of curvature.

EXPT. 25.—Place a small pointed gas flame (p. 54) in front of the convex surface, and observe the virtual erect image formed by reflection at the surface. Between the flame and the surface, place a sheet of thin plate-glass, and adjust this so that there is no parallax between the image of the flame reflected in it, and the image of the flame reflected in the convex surface. Then the image reflected in the convex surface is as far behind the plate-glass, as the flame is in front of the latter. Determine the distance, $v$, from the convex surface to the virtual erect image reflected therein. Then the distance, $u$, from the convex surface to the flame, can be measured directly. If $f$ is the focal length, and $r$ the radius of curvature, of the convex surface—

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r} = \frac{1}{f}.$$

(Remember that $u$ is positive, while $v$ is negative.)

**Curvature of Surfaces of Long-Focus Lens.**—The surfaces of a long-focus lens are nearly plane, and the methods described above do not give accurate results, except in the case of a concave surface. The radius of curvature of the convex surface of a long focus concavo-convex, plano-convex, or bi-convex lens, can best be determined as follows :—

EXPT. 26.—Place the lens in front of an illuminated aperture in a white screen, the convex surface to be examined being turned away from the aperture. Adjust the position of the lens so that the light reflected internally from the convex surface forms an image on the screen beside the aperture.

If $u$ is the distance of the lens from the screen, while $f$ is

the focal length of the lens, the radius of curvature, $r_2$, of the convex surface, is given (p. 78) by the equation—

$$r_2 = uf/(u + f).$$

A simple proof of the formula used above may be of use to the student. When rays from the middle of the aperture are internally reflected at

Fig. 61*a*. Fig. 61*b*.

Rays internally reflected from the convex surface of a convergent lens.

the back surface of the lens, and form an image on the screen, the reflection must take place normally; in other words, the **path of such a ray inside the lens must coincide with part of a radius of the back surface of the lens,** and if this ray were produced backwards it would cross the axis at the centre of curvature C of the back surface (Figs. 61*a*, 61*b*, and 61*c*). Where the ray from the middle of the aperture crosses the *front* surface of the lens, it is bent towards the axis when the front surface is convex (Fig. 61*a*), or when the front surface is concave, if the lens as a whole is convergent (Fig. 61*b*). The ray is bent away from the axis when the front surface is concave, the lens as a whole being divergent (Fig. 61*c*). Thus, when the lens is convergent, C must be further from the lens than the aperture A; when the lens is divergent, C must be between the lens and the aperture. The rays which pass through the back surface of the lens do so without being refracted, since they are incident normally on that surface; thus C will be the position of the virtual image of the aperture that would be seen if one looked through the lens in the direction from

Fig. 61*c*.

Rays internally reflected from the convex surface of a divergent lens.

L to A. Hence, A is the position of an object which gives rise to a virtual image at C, and therefore if LC $= v$, while LA $= u$, we have:—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

where $f$ is the focal length of the lens. Also $v = r_2$, the radius of the back surface of the lens; thus

$$\frac{1}{r_2} = \frac{1}{u} + \frac{1}{f} = \frac{f+u}{uf}$$

$$\therefore \; r_2 = uf / (u + f).$$

When the front surface of the lens is concave (Figs. 61*b* and 61*c*), images can be formed on the screen by rays reflected from either the front or the back surface. If the back surface of the lens be breathed upon, the small drops of water deposited on it will prevent regular reflection. Therefore, if the image formed on the screen is due to reflection from the back surface, breathing on that surface will cause the image to disappear; no appreciable effect will be produced if the image is formed by rays reflected at the front surface.

**Detection of a Plane Surface.**—When it is suspected that one surface of a lens is plane, this suspicion can be readily verified or disproved by the following method:—

EXPT. 27.—Place the surface to be tested on a level with, and near to, the eye, and observe the image of a vertical straight line (such as the edge of a window), formed in it by reflection at grazing incidence. If the surface is plane, the image will be straight and undistorted. Distortion of the image proves that the surface is curved and the method of Expt. 26, or that of Expt. 21, can be used to determine its curvature.

**The Spherometer.**—The curvature of a surface may be measured by purely mechanical means, by the aid of an instrument termed the **Spherometer**. This consists of a rigid metal framework, provided with three pointed legs fixed at the corners of an equilateral triangle, while a fourth leg, equidistant from the other three, can be raised or lowered by means of a fine screw (Fig. 62). Thus, the fourth leg passes through the centre of the circle which may be drawn through the three outer legs. When the central leg is sufficiently raised, the three outer legs can stand firmly on a surface of any shape, and the instrument cannot then be readily rotated. But when the central leg touches the surface, the instrument can be rotated

about the point of contact. This gives us a very delicate means of detecting the instant at which the central leg touches the surface. The central leg has a screw, generally of half a millimetre pitch, cut on it, and is provided with a disc, A, graduated in 50 larger divisions, each of which is again divided into 10 parts. Thus, a complete rotation of the disc raises or lowers the central leg by half a millimetre, and a rotation through one of the larger divisions raises it through $\frac{1}{50} \times \frac{1}{2} = \frac{1}{100}$ mm. At the side of the disc is a vertical scale graduated in half-millimetres, so that the number of complete rotations can be readily ascertained.

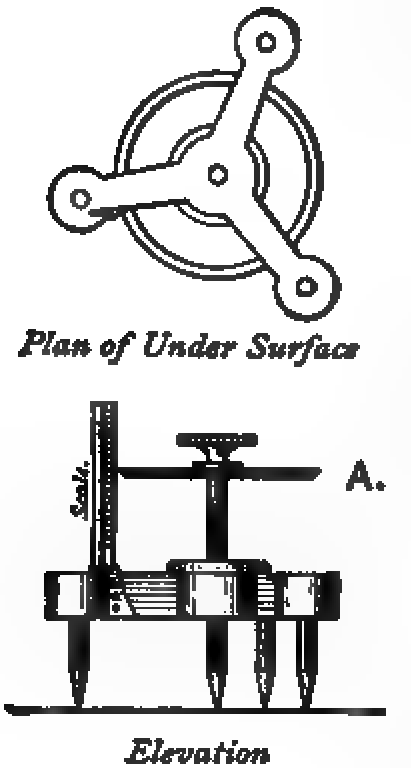



FIG. 62.—The Spherometer. (From Earl's *Physical Measurements*.)

To test the adjustment of the zero, the instrument is placed on a plane surface, such as that of a piece of good plate-glass, and the central leg is adjusted so that lowering it by the smallest amount renders the instrument capable of rotation. The extremities of all four legs are then in a plane, and the zero graduation on the disc should be opposite the edge of the vertical scale.

To determine the curvature of a spherical surface, the spherometer is placed on the surface, and the central leg is adjusted until it just makes contact.

Let ACE (Fig. 63) be a section of the surface by a plane passing through the centre of curvature, K, so that ACEF forms the complete section of the sphere, of which the surface

FIG. 63.—Illustrates the Method of using the Spherometer.

forms a part. Let the central leg of the spherometer touch the surface at C, and draw the diameter CF. The outer legs of the spherometer touch the surface in a plane AE perpendicular to the paper, and the distance DC of the extremity of the central leg above this plane can be determined from a reading of the disc and scale. Let $DC=h$, measured in centimetres. Then, if the distance from the extremity of the central leg to that of one of the outer legs, when all are in a plane, is equal to $d$ centimetres, we have $DA=d$. Let $r$ be the radius of curvature of the surface. Then, from a well-known property of the circle—

$$CD \times DF = (DA)^2, \quad \text{or } h(2r - h) = d^2.$$

$$\therefore r = \frac{d^2 + h^2}{2h} = \frac{d^2}{2h} + h.$$

When $h$ is very small, as will be the case unless the radius of curvature is very small, we may neglect $h$ on the right-hand side of this equation, when—

$$r = d^2/2h.$$

The quantities $d$ and $h$ must both be expressed in terms of the same unit of length, and $r$ is then obtained in terms of this unit.

When the surface is concave, the procedure is similar, except that $h$ then represents the distance of the extremity of the central leg *below* the plane passing through the extremities of the remaining legs.

### Questions on Chapter V—Practical

1. Find the focal length of a concave mirror.

2. Find the focal length of a concave lens by the aid of a convex lens.

3. Determine, by purely optical methods, the refractive index of the glass and the radii of the surfaces of a lens.

4. Measure, with the spherometer, the curvatures of the faces of the given lens. Ascertain also its focal length, and calculate the refractive index of the glass of which it is made.

# CHAPTER VI

## SPHERICAL ABERRATION AND ALLIED PHENOMENA

**Introductory.**—When applying the laws of reflection and refraction to mirrors and lenses, we assumed that the rays from the object, and those which formed the image, lay near to the axis, and were only slightly inclined to the latter. The results obtained are therefore true only when the aperture of the mirror or lens is small, and the object subtends only a small angle at the pole. In the present chapter we must investigate the reflection and refraction of light by mirrors and lenses of wide apertures, and determine the defects produced in the image when the object subtends a considerable angle at the pole.

The image formed by reflection at a plane surface is always a perfect reproduction of the object, since in this case we found it unnecessary to make any assumptions as to the aperture of the reflecting surface. We may therefore, at present, confine our attention to the reflection of light at spherical surfaces, and the refraction of light at plane and spherical surfaces.

**Caustic formed by Reflection at a Spherical Surface.**—Let APB (Fig. 64) be a principal section of a hemispherical mirror, of which C is the centre of curvature, P is the pole, and PQ is the axis. Let O be a luminous point on the axis. Having given a ray from O, incident, in the plane of the paper, at any point on the mirror, it is easy to determine the corresponding reflected ray. Draw the radius passing through the point of incidence, and from the latter point draw a line such that it and the incident ray lie on opposite sides of, and make equal angles with, the radius. The line so drawn represents the reflected ray. In Fig. 64, a number of rays, reflected from various

points on the hemispherical surface, have been drawn in this manner.

It will be noticed that the rays, reflected from points in the immediate neighbourhood of the pole P, intersect the axis and each other in a point I; this is the point which we have termed the **image of** O. On the other hand, it will be observed that when the point at which reflection occurs is remote from the pole, the reflected ray cuts the axis at a point nearer to the mirror than I; and the more remote from the pole is the point of reflection, the nearer to the mirror does the reflected ray cut the axis.

FIG. 64.—Caustic formed by Reflection at a Spherical Surface.

Similarly, when rays parallel to the axis are reflected from a spherical mirror, those rays alone which are reflected near the pole pass through the principal focus; those reflected near the periphery of the mirror cut the axis at a point nearer to the mirror than the principal focus. This phenomenon is termed **spherical aberration.** The distance from the principal focus to the point where the axis is cut by rays reflected from the periphery of the mirror (the incident rays being parallel to the axis) is termed **the spherical aberration of the mirror.**

It must also be noticed that any two rays, reflected from neighbouring points of the mirror at a distance from the pole, intersect each other before reaching the axis. Each point of intersection of neighbouring reflected rays is a sort of focus, and if we draw a curve joining all such foci, it is obvious that all of the reflected rays touch this curve. Such a curve is drawn in the lower half of Fig. 64; it is termed the **caustic curve.** The image I is situated at the cusp of the caustic curve.

If we imagine Fig. 64 to rotate about the axis PQ, the

semicircle APB will describe the hemispherical reflecting surface, and the caustic curve will describe a surface termed the **caustic surface.** All of the rays reflected from the hemispherical mirror touch the caustic surface.

When a beam of sunlight shines into a cup containing tea, &c., the caustic curve, due to the rays reflected from the side of the cup, is formed on the surface of the tea.

EXPT. 28.—Obtain a circular glass ring, cut from a cylindrical glass shade, and divide this into two semicircular portions. Lay one of these on a sheet of white paper, and place a small electric glow lamp at a point on the axis. A bright line on the paper indicates the position of the caustic curve. Notice the form of this curve when the lamp is placed at the principal focus, *i.e.*, half way between the centre of curvature and the pole.

EXPT. 29.—In a manner similar to that used in drawing Fig. 64, draw the caustic curve when the incident rays radiate from the principal focus.

EXPT. 30.—In a similar manner, determine the form of the *virtual* caustic curve, when light-rays are reflected from a convex surface.

**Focal Lines formed by Reflection at a Spherical Surface.**—Let APB (Fig. 65) be part of a principal section of a hemispherical surface, of which C is the centre of curvature, P is the pole, and PQ is the axis. Let O be a luminous point on the axis, and let OE, OD, be rays incident, in the plane of the paper, at neighbouring points, E, D, of the surface. The corresponding reflected rays, EH, DK, intersect each other at $F_1$, and cross the axis at different points. If OE, OD, are the limits of the section, by the plane of the paper, of a divergent incident pencil, then EH, DK, form the limits of the section, by the plane of the paper, of the reflected pencil.

FIG. 65.—Focal Lines formed by Reflection at a Spherical Surface.

We must now enquire into the nature of the complete

reflected pencil. If we suppose Fig. 65 to rotate through a small angle about the axis PQ, the curved line DE describes a small element of the reflecting surface, and the space DOE describes the pencil diverging from O incident on this element. Similarly, the space EHKD describes the pencil reflected from the element. The point $F_1$ describes a short, approximately straight, line, perpendicular to the plane of the paper; every ray in the reflected pencil passes through this line, which is termed the **First Focal Line** of the reflected pencil. Every ray also passes through the axis, so that the transverse section of the reflected pencil, where it crosses the axis of the mirror, is a short straight line, $F_2$, in the plane of the paper; this is termed the **Second Focal Line** of the reflected pencil. Thus, as we pass from $F_1$ to $F_2$, the transverse section of the pencil changes from a straight line *perpendicular* to the plane of the paper, to a straight line *in* the plane of the paper; and as this change proceeds continuously, it follows that the width of the pencil perpendicular to the plane of the paper decreases, as the width of the pencil in the plane of the paper increases. Consequently, somewhere between $F_1$ and $F_2$, the pencil must be equally wide in both directions. Here, the section of the pencil is approximately circular, and is termed the **Circle of Least Confusion**; this is the nearest approach to a point focus in the reflected pencil.

Since a reflected pencil, formed in the manner just described, does not anywhere pass through a point, it is termed an **astigmatic pencil**.[1]

From the centre of curvature, C, draw the radius CG, passing through the middle point of the element DE. Then the line CG forms the principal axis of the surface element CD, and O becomes a point source of light situated at a distance from the axis. The pencil OED may now be termed an **oblique centric pencil**: *oblique*, since its constituent rays are considerably inclined to the axis; and *centric*, since the rays are incident in the immediate neighbourhood of the middle point of the element. We thus see that **an oblique centric pencil gives rise, on reflection at a spherical surface, to an astigmatic pencil, which passes through two mutually perpendicular focal lines, $F_1$ and $F_2$. Between these two focal lines is the circle of least confusion.**

[1] Greek, *a*, without, and *stigma*, a spot or point.

The circle of least confusion is the nearest approach to an image of the point source O, which can be formed by oblique centric reflection. Measurement shows that the circle of least confusion is nearer to the pole of the element DE, than I′, which is the image of a point source on the axis CG, at a distance equal to DO from the element. This result may be generalised by stating that for oblique centric reflection a mirror acts as if its focal length were numerically less than for direct reflection.

It will now be understood that the construction used on p. 36, in order to determine the image corresponding to an object of finite dimensions, will give correct results only when the object subtends a small angle at the pole of the mirror. In that case, a pencil from an extremity of the object is incident at a very small angle, and the two focal lines of the reflected pencil practically coincide. When the object is large, the image will be curved, its edges being drawn up toward the mirror, and it will also be confused, owing to the circumstance that a point near the edge of the object gives rise to a relatively large circle of least confusion in the reflected pencil.

**Caustic formed by Refraction at a Spherical Surface.**—Let APB (Fig. 66) be a principal section of a hemispherical surface

FIG. 66.—Caustic formed by Refraction at a Spherical Surface.

of which C is the centre of curvature, P is the pole, and PC (produced) is the axis. Let there be air to the right, and glass ($\mu = 1{\cdot}5$) to the left, of the surface. Having given a ray incident in the plane of the paper on any point of the surface, the refracted ray may be determined by the graphical method explained on p. 46. In Fig. 66, where the incident rays are

supposed to be parallel to the axis, several refracted rays are drawn. It will be seen that the rays, refracted near the pole of the surface, cut the axis and each other in a point, F. This point is the **second principal focus** of the surface. On the other hand, rays refracted at points of the surface remote from the pole, cut the axis at points nearer to the surface than the second principal focus; and the more remote from the pole is the point of refraction, the nearer to the surface does the refracted ray cut the axis. This phenomenon is termed **spherical aberration.** The distance from the second principal focus to the point on the axis cut by rays refracted at the periphery of the surface, (the incident rays being parallel to the axis), is termed **the spherical aberration of the surface.**

It will also be noted that rays, refracted at neighbouring points of the surface remote from the pole, intersect each other before reaching the axis. Each point of intersection of neighbouring refracted rays is a sort of focus, and a curve joining all such foci is termed the **Caustic Curve.** The caustic curve is drawn by itself in the lower part of Fig. 66.

EXPT. 31.—Obtain the virtual caustic curve when parallel rays are refracted at a concave hemispherical surface.

EXPT. 32.—Obtain the virtual caustic curve when rays, diverging from a point, are refracted at a plane surface.

**Focal Lines formed by Refraction at a Spherical Surface.**—Fig. 67 is similar to Fig. 66, except that only two rays incident at points remote from the pole P, together with the corresponding refracted rays, are shown. The refracted rays cut each other at $F_1$, and cross the axis at different points. By supposing the diagram to rotate through a small angle, we find that

FIG. 67.—Focal Lines formed by Refraction at a Spherical Surface.

a parallel pencil, incident at D, gives rise to an astigmatic refracted pencil, passing through two mutually perpendicular lines at $F_1$ and $F_2$ respectively. $F_1$ is **the first focal line** of the refracted pencil; $F_2$, which forms the transverse section of the refracted pencil where it crosses the axis, is **the second focal line** of the refracted pencil. From reasoning similar to that used on p. 125, it follows that, somewhere between $F_1$ and $F_2$, the section of the refracted pencil is approximately circular. This circular section is termed the **circle of least confusion**, and forms the nearest approach to a point focus to be met with in the refracted pencil.

If we draw the radius CD, then this line is the axis of the surface element immediately surrounding D. The incident pencil now becomes an **oblique centric pencil** (p. 125), and we see that this gives rise to an astigmatic refracted pencil. Further, since Fig. 67 is drawn on the supposition that $\mu = 1{\cdot}5$, it follows that the second principal focus for direct axial pencils is at a distance $\mu r/(\mu - 1) = 1{\cdot}5r/0{\cdot}5 = 3r$ from the pole. Direct measurement will show that the distance, $DF_2$, is considerably less than $3 \times DC$; and as the circle of least confusion lies between $F_1$ and $F_2$, it follows that **a refracting surface acts as if its focal lengths were numerically smaller for oblique centric pencils than for direct axial pencils.**

**Image produced by Oblique Centric Refraction through a Lens.**—Let us suppose that an object subtends a considerable angle at a lens; and that, by means of a diaphragm with a small circular aperture, the transmitted rays are limited to those passing through the middle of the lens. Then the extremities of the image remote from the axis will be formed by oblique centric pencils (Fig. 68). As proved above, the greater the obliquity of an incident centric pencil, the shorter is the effective focal length of each surface of the lens for that pencil, and therefore, the shorter is the focal length of the lens as a whole for the pencil.

When a real image is formed by a convergent lens, the shorter the focal length of the lens, the nearer to the lens is the image. Thus, the parts of the image remote from the axis will be drawn up toward the lens, so that the image as a whole is curved (Fig. 68, I.).

When a virtual magnified image is formed by a convergent lens, the shorter the focal length of the lens, the farther from the lens is the

image. Thus, the parts of the image remote from the axis are drawn away from the lens, and the image is curved (Fig. 68, II.). It is perhaps easiest to remember that **the image formed by oblique centric pencils, refracted through a convergent lens, always has a positive radius of curvature.**

In the case of a divergent lens, the shorter the focal length, the nearer is the image to the lens. Thus, the parts of the image remote from the axis are drawn up toward the lens (Fig. 68, III.). In this case **the radius of curvature of the image is negative.**

When a virtual image is formed, and this is viewed directly by the eye, curvature of the image is of small moment. The angles subtended at the eye by different parts of the image are proportional to the angles subtended by the corresponding parts of the object. On the other hand, when a real image is formed on a flat screen, it is obvious that, if curvature exists, all parts of the image cannot be equally sharply focussed.

FIG. 68.—Curvature of Images formed by Oblique Centric Pencils.

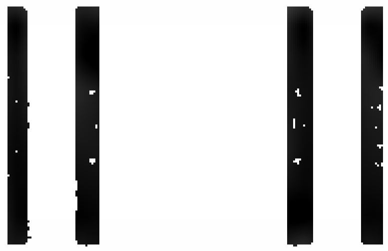
**Curvature and Distortion of Image formed by Excentric Pencils.**—An excentric pencil is one which is refracted through a lens at a point remote from the pole. The general characteristics of an image formed by such pencils must now be investigated. An excentric ray parallel to the axis, when refracted at a spherical surface, cuts the axis at a point closer to the surface than the principal focus (p. 127). Similarly, an excentric ray parallel to the axis, when refracted through a *lens*, cuts the axis at a point nearer to the lens than the principal focus. In other words, we may say that, for excentric rays, the principal foci of a lens are nearer to the lens than the true foci for direct centric rays. Let PQ (Fig. 69) be the

principal plane of a lens of wide aperture, and let OBA be an object placed nearer to the lens than the first principal focus. The ray from the extremity A of the object, which is rendered parallel to the axis after refraction through the lens, cuts the lens near the periphery, and is therefore an excentric ray; consequently, it proceeds from a point on the axis nearer to the lens than the first principal focus. Similarly, a ray from A parallel to the axis will proceed, after refraction through the lens, toward a point closer to the lens than the second principal focus. These considerations will enable us to obtain the point C, which is the image of A formed by excentric rays. Let B be the middle point of the object; to determine the image of B, we must find rays from B, corresponding to those found above with respect to A. The rays from B will be less excentric than those from A, and will consequently cut the axis at points farther from the lens, and nearer to the principal foci, than the corresponding rays from A. Remembering this, the construction for the point D, which is the image of B, will be readily understood. It at once becomes evident that **the image is curved,** its radius of curvature being positive; in addition, **the image is distorted,** the part DC being more magnified than the part ID. The curvature is not of much importance, but the distortion may render the image very unlike the object, the peripheral parts being magnified to a greater extent than the central parts. Thus, if the object is ruled as in Fig. 70, A, so as to consist of a number of equal squares, its virtual image, seen through a convergent lens, will have the character of B (Fig. 70).

FIG. 69.—Curvature and Distortion of Virtual Image, formed by Convergent Lens of Wide Aperture.

EXPT. 33.—Look at a piece of squared paper through a convergent lens of about 1 or 2 inches focus. The image seen will resemble B (Fig. 70).

The characteristics of a real image, formed by excentric pencils refracted through a lens of wide aperture, can be understood from Fig. 71. Let PQ be the principal plane of the lens, and let OBA be the object, of which B is the middle point.



A. B. C.

FIG. 70.—Different Kinds of Distortion produced by a Convergent Lens.

A narrow excentric pencil from A is brought to an approximate focus at C, while a similar pencil from B is brought to a focus at D, the point C being nearer to the lens than D. Thus, **the image is curved**, its radius of curvature being positive; in addition **the image is distorted**, the part DC being smaller than the part ID. Accordingly, if the object resembles A (Fig. 70), the image will resemble C, the parts of the image remote from the axis being compressed. When a single lens is used as a photographic objective, this distortion is very marked; straight lines, such as the edges of buildings, give rise to curved lines on the negative.

FIG. 71.—Curvature and Distortion of Real Image, formed by Convergent Lens of Wide Aperture.

We can now form a general idea of the defects of an image produced, in any given circumstances, by a convergent lens.

*When the transmitted rays are limited by a diaphragm with a small central aperture.*—The image, real or virtual, is curved, its radius of curvature being positive. Distortion is absent. The image is distinct.

*When no diaphragm is used, but the object is itself an image, so that*

*the pencils by which the final image is formed are narrow.*—In this case each pencil is refracted through a limited part of the lens. The image is curved, its radius of curvature being positive; it is also distorted, the peripheral parts being disproportionately compressed (real image) or expanded (virtual image). Each part of the image is distinct, being formed by a single narrow pencil.

*When the object is real, and no diaphragm is used.*—Each part of the image is confused, being formed by pencils refracted through various parts of the lens. The image is also curved and distorted.

**Methods of minimising the Effects of Spherical Aberration.**—The distortion due to spherical aberration can be diminished by decreasing the effective aperture of the lens, by means of a diaphragm with a central aperture of suitable size. Such a diaphragm is termed a **stop**. When a sharply-focussed real image is required, it is preferable to stop out the *central portion* of the lens, and form the image by pencils refracted through the marginal portions. This was first pointed out by Lord Rayleigh; the reason will be explained in the chapter on Diffraction (Chap. XVII).

It is possible to design a lens of a given focal length which shall produce the minimum amount of spherical aberration. The train of reasoning which must guide us in this task is as follows. It will be seen from Fig. 72, that **the rays refracted at a spherical surface cut the axis in a single point, so long as their deviation remains small.** Further, the aberration increases more and more quickly as the deviation becomes larger. Consequently, it must be our aim to confine the deviation produced at each surface within the narrowest possible limits. The best result will be obtained when each ray receives equal increments of deviation at the two surfaces of the lens; in any other arrangement, the diminished aberration at the surface producing the smaller deviation is more than counterbalanced by the increased aberration at the other surface. For the deviations to be equal, light must enter one surface of the lens at an angle equal to that at which it emerges from the other surface; or **the angles of incidence and emergence must be equal.** It will be remembered that a similar condition must be complied with in order that an object shall be seen distinctly through a prism (p. 59).

Let us now suppose that a lens is required to bring parallel rays to a focus at a given point on the axis. The focal length of

the lens then becomes known, but we have yet to decide whether the lens shall be bi-convex, plano-convex, or concavo-convex.

A glance at Fig. 72 will show that with an equi-convex lens, the angles of incidence and emergence of the rays cannot be equal. If the lens is plano-convex, and the plane surface faces the incident parallel rays, the whole of the required deviation must be produced at the second (convex) surface. If, on the other hand, the convex surface faces the parallel rays, both surfaces produce nearly equal deviations, and the spherical aberration will be minimised. This accounts for the frequent use of plano-convex lenses in optical instruments. **The convex surface should face toward the incident rays, or the emergent rays, whichever are the more nearly parallel to the axis.**

FIG. 72.—Illustrates the Method of minimising Spherical Aberration.

When powerful spectacles are worn, the lenses are generally plano-convex, the plane side facing the eye. The more convex surface of a telescope objective always faces toward the incident rays. The spherical aberration of a plano-convex lens, when the plane surface faces the incident parallel rays, is between four and five times as great as when the convex surface faces the incident rays.

**Crossed Lens.**—For a lens of crown glass ($\mu = 1 \cdot 5$) to produce absolutely the *smallest possible* amount of aberration, it should be bi-convex, the radii of curvature of its surfaces being in the ratio 1 : 6, the more strongly curved surface facing the incident parallel rays. Such a lens is termed a **crossed lens**. The spherical aberration of such a lens is, however, only about 2 or 3 per cent. smaller than that of a plano-convex lens, used as above described. Hence, crossed lenses of crown glass are seldom used. For flint glass ($\mu = 1 \cdot 6$), the crossed lens is plano-convex.

**Chromatic Effects of Spherical Aberration.**—Since the refractive index of a lens is greater for blue than for red rays, it follows that the deviation produced at each surface of a lens is greater for blue than for red rays. Consequently the spherical aberration of the lens is greater for blue than for red rays. If the lens is used to form a real image, the marginal parts of the blue image will be compressed to a greater extent than those of the red image. If the lens is convergent, and a virtual erect image is formed by it, the marginal parts of the blue image will be distended to a greater extent than those of the red image.

The chromatic effects of spherical aberration cannot be completely eliminated, but they can be diminished by equalising the deviations produced at the two surfaces of the lens, or by producing the required resultant deviation by means of two lenses.

## Questions on Chapter VI

1. What is meant by a caustic curve? A luminous point is moved from a distance along the axis of a concave mirror towards the mirror. Indicate the position of the geometrical focus, and the form of the caustic for various positions of the point.

2. Show that a small pencil of rays obliquely reflected from a spherical surface passes approximately through two small straight lines lying in planes at right angles to each other.

3. A source of light is placed at the principal focus of a concave mirror of large aperture. Draw a picture showing the paths of the reflected rays.

4. A small cone of rays from a point source is incident obliquely upon a plane transparent surface. Show that after refraction the rays appear to diverge from two focal lines.

5. Investigate the form of the image of a small straight line placed on, and perpendicular to, the axis of a lens.

6. Discuss the nature of the distortion produced when an object is viewed through (1) a convex, (2) a concave lens.

# CHAPTER VII

## REFRACTION OF AXIAL PENCILS BY A THICK LENS

WHEN investigating the refraction of axial pencils by a thin lens (p. 67), we assumed that the distance $v'$, from the first surface of the lens to the image formed by refraction at that surface, was very great in comparison with $t$, the thickness of the lens. The image formed by refraction at the first surface acts as the object in the refraction at the second surface; and we assumed that $(v' + t)$, the distance of that object from the second surface, was sensibly equal to $v'$. In thick, short focus lenses, this assumption is not generally admissible. In the present chapter we shall investigate the refraction of axial pencils by a lens of sensible thickness.

Let $u$ be the distance, measured from the first surface of the lens, to a luminous point on the axis; and let an image of this point be produced by the first refraction, at a distance $v'$ from the first surface. Then, if $\mu$ is the index of refraction of the substance of which the lens is composed, we have (p. 62)—

$$\frac{\mu}{v'} - \frac{1}{u} = -\frac{1}{f_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f_1$ is the value of $u$ corresponding to $v' = \infty$. It is obvious that $f_1$ will be the *first focal distance of the first surface of the lens* (p. 62); and if $r_1$ is the radius of curvature of that surface, we shall have—

$$\frac{1}{f_1} = -\frac{\mu - 1}{r_1}.$$

If $v$ is the distance, measured from the second surface of the

lens, to the image formed by the refraction at that surface, then we shall have—

$$\frac{1}{v} - \frac{\mu}{v' + t} = \frac{1}{f_2}, \quad \ldots\ldots\ldots \quad (2)$$

where $f_2$ is the value of $v$ corresponding to $v' = \infty$. $f_2$ obviously will be equal to the *second focal distance of the second surface* (p. 63), and its value will be given by

$$\frac{1}{f_2} = \frac{1 - \mu}{r_2},$$

where $r_2$ is the radius of curvature of the second surface.

Equating the values of $v'$ found from (1) and (2)—

$$\frac{\mu u f_1}{f_1 - u} = -t + \frac{\mu v f_2}{f_2 - v}.$$

Multiplying throughout by $(f_1 - u)(f_2 - v)$—

$$\mu u f_1(f_2 - v) + t(f_1 - u)(f_2 - v) - \mu v f_2(f_1 - u) = 0.$$

Collecting terms in $uv$, $u$, and $v$,

$$\{\mu(f_2 - f_1) + t\}\, uv + (\mu f_1 f_2 - f_2 t)u - (\mu f_1 f_2 + f_1 t)v + f_1 f_2 t = 0.$$

$$\therefore\ uv + \frac{\mu f_1 f_2 - f_2 t}{\mu(f_2 - f_1) + t}u - \frac{\mu f_1 f_2 + f_1 t}{\mu(f_2 - f_1) + t}v + \frac{f_1 f_2 t}{\mu(f_2 - f_1) + t} = 0. \quad (3)$$

This equation can be transformed into one of the form—

$$\frac{1}{v - \beta} - \frac{1}{u - \alpha} = \frac{1}{F}, \quad \ldots\ldots\ldots \quad (4)$$

For, if we multiply (4) throughout by $F(u - \alpha)(v - \beta)$, and collect terms in $uv$, $u$, and $v$, we obtain—

$$uv - (F + \beta)u + (F - \alpha)v - F(\beta - \alpha) + \alpha\beta = 0. \quad (5)$$

In order that (3) and (5) shall be identical, we must have—

$$-F - \beta = \frac{\mu f_1 f_2 - f_2 t}{\mu(f_2 - f_1) + t} \quad \ldots\ldots\ldots \quad (6)$$

$$-F + \alpha = \frac{\mu f_1 f_2 + f_1 t}{\mu(f_2 - f_1) + t} \quad \ldots\ldots\ldots \quad (7)$$

$$\alpha\beta - F\beta + F\alpha = \frac{f_1 f_2 t}{\mu(f_2 - f_1) + t} \quad \ldots\ldots\ldots \quad (8)$$

(6), (7), and (8) are three simultaneous equations to determine the three unknown quantities, $\alpha$, $\beta$, and $F$. To solve these, first multiply

together the corresponding sides of (6) and (7). From this operation we obtain—

$$-\alpha\beta + F\beta - F\alpha + F^2 = \frac{(\mu f_1 f_2 - f_2 t)(\mu f_1 f_2 + f_1 t)}{\{\mu(f_2 - f_1) + t\}^2} \quad . \quad . \quad (9)$$

Adding (8) and (9)—

$$F^2 = \frac{(\mu f_1 f_2)^2 + \mu f_1 f_2 t(f_1 - f_2) - f_1 f_2 t^2}{\{\mu(f_2 - f_1) + t\}^2} + \frac{f_1 f_2 t}{\mu(f_2 - f_1) + t}$$

$$= \frac{(\mu f_1 f_2)^2 - f_1 f_2 t\{\mu(f_2 - f_1) + t\} + f_1 f_2 t\{\mu(f_2 - f_1) + t}{\{\mu(f_2 - f_1) + t\}^2}$$

$$= \frac{(\mu f_1 f_2)^2}{\{\mu(f_2 - f_1) + t\}^2}$$

$$\therefore \; F = \pm \frac{\mu f_1 f_2}{\mu(f_2 - f_1) + t}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In order to determine whether the positive or negative sign applies to the present problem, it should be noticed that (10) must be true for all values of $t$. When $t = 0$, the problem reduces to the determination of the relative positions of object and image, when the light is refracted by a thin lens. Now, since rays from a luminous point at a distance $f_1$ from the first surface, are reduced to parallelism within the lens, and these parallel rays are brought to a focus at a point distant $f_2$ from the second surface of the lens, it is evident that $f_1$ and $f_2$ are conjugate focal distances. When $t = 0$, $\alpha$ and $\beta$ in (4) will both be equal to zero. Substituting $\alpha = 0, u = f_1, \beta = 0, v = f_2$ in (4), we obtain—

$$\frac{1}{f_2} - \frac{1}{f_1} = \frac{1}{F},$$

$$\therefore \; F = -\frac{f_1 f_2}{f_2 - f_1}.$$

Substituting $t = 0$ in (10), we see that we obtain the value of F just found, provided that we take the negative sign. Hence,

$$F = -\frac{\mu f_1 f_2}{\mu(f_2 - f_1) + t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From (11) and (6), it is easily seen that—

$$\beta = \frac{f_2 t}{\mu(f_2 - f_1) + t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (11) and (7), we see that—

$$a = \frac{f_1 t}{\mu (f_2 - f_1) + t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, $a f_2 = \beta f_1$, an important relation which we shall subsequently find useful.

It should be noted that, if we measure V from a point at a distance $\beta$ in the positive direction from the second surface, while U is measured from a point at a distance $a$ in the positive direction from the first surface, then we may write—

$$u - a = \text{U}, \text{ and } v - \beta = \text{V}.$$

We can now rewrite (4)—

$$\frac{1}{\text{V}} - \frac{1}{\text{U}} = \frac{1}{\text{F}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

a formula similar to that used in connection with a thin lens (p. 69).

**Principal Points and Principal Planes.**—The point at a distance $a$ from the first surface of the lens is termed the **first principal point of the lens.** The point at a distance $\beta$ from the second surface is termed the **second principal point of the lens.** When positive, these distances are measured from the surfaces of the lens in a direction opposite to that of the incident light.

Planes drawn perpendicular to the axis through the principal points are termed the **principal planes of the lens.**

$\text{U} = -\text{F}$ when $\text{V} = \infty$. Hence, $-\text{F}$ is the **first focal distance** of the lens. Its magnitude is, of course, determined by (11), and it is measured from the first principal point of the lens. Also, $\text{V} = \text{F}$ when $\text{U} = \infty$. Hence, F will be the **second focal distance** of the lens; this is measured from the second principal point of the lens. The first and second focal distances of a lens *surrounded by a medium of uniform refractive index*, will, therefore, be equal in magnitude but opposite in sign.

**Refraction by a Lens separating Media of Different Refractive Indices.**—Let us suppose that the refractive index of the substance from which a lens is formed is equal to $\mu_2$, and that the refractive indices of the media on the positive and

negative sides of the lens are respectively equal to $\mu_1$ and $\mu_3$. Then, for the first refraction—

$$\frac{\frac{\mu_2}{\mu_1}}{v'} - \frac{1}{u} = -\frac{1}{f_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where
$$\frac{1}{f_1} = -\frac{\mu_2 - \mu_1}{\mu_1} \cdot \frac{1}{r_1}.$$

For the second refraction, we shall have—

$$\frac{\frac{\mu_3}{\mu_2}}{v} - \frac{1}{v' + t} = \frac{\frac{\mu_3}{\mu_2}}{f_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

where
$$\frac{1}{f_2} = \frac{\mu_3 - \mu_2}{\mu_3} \cdot \frac{1}{r_2}.$$

We may transform $(a)$ and $(b)$ into forms similar to (1) and (2), pp. 135–6,

$$\frac{\mu_2}{v'} - \frac{1}{\frac{u}{\mu_1}} = -\frac{1}{\frac{f_1}{\mu_1}}$$

$$\frac{1}{\frac{v}{\mu_3}} - \frac{\mu_2}{(v' + t)} = \frac{1}{\frac{f_2}{\mu_3}}.$$

These two equations are obtained from (1) and (2), by writing $\mu_2$ instead of $\mu$, $\frac{u}{\mu_1}$ instead of $u$, $\frac{v}{\mu_3}$ instead of $v$, $\frac{f_1}{\mu_1}$ instead of $f_1$, and $\frac{f_2}{\mu_3}$ instead of $f_2$. The solution, which consists essentially in the elimination of $v'$, will be obtained by making these substitutions in (4), (11), (12), and (13). Thus (4) must be altered into—

$$\frac{1}{\frac{v}{\mu_3} - \beta} - \frac{1}{\frac{u}{\mu_1} - a} = \frac{1}{F}.$$

$$\therefore \frac{\mu_3}{v - \mu_3\beta} - \frac{\mu_1}{u - \mu_1 a} = \frac{1}{F}.$$

If $U = u - \mu_1 a$, and $V = v - \mu_3\beta$, we may rewrite this equation—

$$\frac{\mu_3}{V} - \frac{\mu_1}{U} = \frac{1}{F}.$$

The first focal distance of the lens will be equal to $-\mu_1 F$, and the second focal distance will be equal to $\mu_3 F$. In this case, therefore, the focal distances of the lens are numerically unequal, being in the ratio of the refractive index of the first, to that of the second medium. The respective distances of the first and second principal points from the first and second surfaces of the lens will be equal to $\alpha' = \mu_1\alpha$ and $\beta' = \mu_3\beta$.

$$\alpha' = \mu_1\alpha = \mu_1 \frac{\dfrac{f_1}{\mu_1}t}{\mu_2\left(\dfrac{f_2}{\mu_3} - \dfrac{f_1}{\mu_1}\right) + t} = \frac{\mu_1\mu_3 f_1 t}{\mu_2(\mu_1 f_2 - \mu_3 f_1) + \mu_1\mu_3 t}.$$

$$\beta' = \mu_3\beta = \frac{\mu_1\mu_3 f_2 t}{\mu_2(\mu_1 f_2 - \mu_3 f_1) + \mu_1\mu_3 t}.$$

It should be noted that $f_2\alpha' = f_1\beta'$.

$$F = -\frac{\mu_2\dfrac{f_1}{\mu_1} \cdot \dfrac{f_2}{\mu_3}}{\mu_2\left(\dfrac{f_2}{\mu_3} - \dfrac{f_1}{\mu_1}\right) + t} = -\frac{\mu_2 f_1 f_2}{\mu_2(\mu_1 f_2 - \mu_3 f_1) + \mu_1\mu_3 t}.$$

**The Principal Points of a Lens are Conjugate Foci.**—We may prove this statement for the case of a lens separating media of differentre fractive indices, and it will then become evident that it applies to a lens under any conditions.

As proved above—

$$\frac{\mu_3}{v - \beta'} - \frac{\mu_1}{u - \alpha'} = \frac{1}{F}.$$

$$\therefore\ v - \beta' = \frac{\mu_3 F(u - \alpha')}{\mu_1 F + (u - \alpha')}.$$

When $u = \alpha'$, we have $(v - \beta') = 0$, whatever may be the value of $\mu_3$, $\mu_1$, and F. Hence, an object at a distance $\alpha'$ from the first surface of the lens (*i.e.* an object at the first principal point) will produce an image at a distance $\beta'$ from the second surface of the lens (*i.e.* at the second principal point). When the first principal point is outside the lens, on the positive side, a real object may be placed there. When, however, the first principal point lies inside, or on the negative side of, the lens, the object must be virtual, formed by a pencil of rays which, before entering the lens, converge toward the first principal point. A real object placed at the first principal point, when that is inside

the lens, would be seen by an eye on the negative side of the lens, after the light had been refracted only at the second surface, while the formulæ deduced above apply to refractions at *both* surfaces.

**The Principal Planes of a Lens are Planes of Unit Magnification.**—Let us suppose that a real image, situated at the position occupied by the first principal plane of the lens, would be formed by convergent pencils of light, if the lens were removed. With the lens in position, this image will no longer be formed if the first principal point is within the lens, since the light is refracted by the first surface of the lens. Before entering the lens, however, the light will converge as if to form the same image as before. The dimensions of this virtual image will naturally be the same as those of the real image that would be produced if the lens were removed.

In these circumstances an eye, placed on the negative side of the lens, will see a virtual image in the second principal plane of the lens. This follows from the fact that the principal points are conjugate foci. It will now be proved that the dimensions of the image seen in the second principal plane are equal to those of the image in the first principal plane, and if one image is erect, then the other is erect also. In other words, the magnification produced by the lens in these circumstances is equal to + 1 (p. 72).

We must first obtain an expression for the magnification produced by a thick lens.

Let an object, O (Fig. 73), be placed on the axis of the lens, and let A be the first focal point of the first surface of the lens. The rays of light radiating from A will be rendered parallel after refraction at the first surface. Let AB be the ray from the upper extremity of the object O which passes through A, the first focal point of the first surface, or which would pass through A if produced backwards. This ray will traverse the lens in the path BC, parallel to the axis. After refraction at the second surface, the same ray will pass through F, the second focal point of the second surface of the lens. From the formulæ already obtained, we can calculate the position of the image, I, ultimately formed; further, it is clear that the ray CF will pass through the point of this image which corresponds to the upper extremity of O. Hence, we can draw the image I.

Drop perpendiculars, BD, CE, from B and C on to the axis. When the rays AB, CF, are nearly parallel to the axis, the feet of these perpendiculars will approximately lie in the intersections of the axis by the surfaces of the lens. Let DB = EC = $y$. Let $u$ = distance of O from first surface, and $v$ = distance of I from second surface; in the figure $u$ = DO (approximately), and $v = -$IE (approximately). If $f_1$ is the first focal distance

FIG. 73.—To prove that the Principal Planes are Planes of Unit Magnification.

of the first surface (= DA), while $f_2$ is the second focal distance of the second surface (= $-$FE), and if the heights of object and image are equal to o and i respectively, then—

$$\frac{y}{f_1} = \frac{o}{f_1 - u}.$$

Also—

$$\frac{y}{-f_2} = \frac{i}{-(f_2 - v)}.$$

Let $m$ = the magnification produced by the lens. Then—

$$m = \frac{i}{o} = \frac{f_1(f_2 - v)}{f_2(f_1 - u)}.$$

This formula will give the magnification in any case; if the value of $m$ is found to be negative, it means that the image is inverted.

If we now suppose the object to lie in the first principal plane, $u = \alpha'$ (p. 140). As already proved, the corresponding image will lie in the second principal plane, so that $v = \beta'$. In this case—

$$m = \frac{f_1 f_2 - f_1\beta'}{f_1 f_2 - f_2\alpha'} = +1,$$

since $f_1\beta' = f_2\alpha'$ (p. 140).

As a consequence, a pencil of rays, converging, before entering the lens, towards a point in the first principal plane, at a distance $h$ from the axis, will, after refraction by the lens, diverge from a point in the second principal plane, on the same side of the axis, and at an equal distance, $h$, from it. Finally, **any ray directed towards a point in the first principal plane, at a distance $h$ from the axis, will give rise to a transmitted ray proceeding from a point in the second principal plane, on the same side of the axis, and at a distance, $h$, from it.**

**Graphic Determination of Images.**—Having determined the principal planes of a lens, we can easily construct the image corresponding to any object placed on the axis of the lens.

Fig. 74.—Graphic Determination of Image.

Let planes be drawn through $P_1$, $P_2$ (Fig. 74), the principal points of a lens, and let $F_1$ and $F_2$ be the first and second principal foci of the lens. Draw any object, O, above the axis. Then the ray from the upper extremity of O, which passes through $F_1$ (after being produced backwards if necessary), will be rendered parallel after refraction by both surfaces of the lens. Also this ray will leave the second principal plane at a point on the same side of the axis, and as far from it, as the point in which the first principal plane is cut by the incident ray. This follows from the law deduced in the last paragraph.

Further, the ray parallel to the axis, from the upper extremity of O, will, on leaving the lens, pass through the second principal focus, $F_2$. Drawing the ray parallel to the axis from the upper extremity of O, the intersection of the first principal plane by this ray will be on the same side of the axis, and

as far from it, as the intersection of the second principal plane by the ray refracted through $F_2$. Hence, we may draw the latter ray. The two refracted rays which we have determined will meet if suitably produced, in the extremity of the image I, which corresponds to the upper extremity of O.

It should be noticed that the ray passing through the lens does not actually pass through the two points, on the same side of the axis and equidistant from it, in the principal planes. The line joining the intersections of the first and second surfaces of the lens, by the incident and refracted rays respectively, will, of course, give the actual path through the lens.

**Formulæ for Magnification.**—A glance at Fig. 74 will suffice to show the truth of the following formulæ. Let $u$ and $v$ be the distances of object and image, measured from the first and second principal points respectively; and let $F_1$ and $F_2$ be the first and second focal distances of the lens, also respectively measured from the first and second principal points. Then—

$$m = \frac{i}{o} = \frac{F_1}{F_1 - u} = \frac{-F_2 + v}{-F_2}.$$

(Remember that in Fig. 74, $F_2$ has a negative value.)

The above formulæ should be compared with those obtained for a thin lens (p. 72). As already proved, $F_1$ and $F_2$ will differ only in sign, unless the media on opposite sides of the lens have unequal refractive indices.

**Nodal Points.**—There are two other points on the axis of a thick lens which possess properties which can sometimes be used to simplify the solution of problems. A ray of light, directed from the positive side of the lens toward the first of these points on the axis, will, after refraction by the lens, proceed from the second point on the axis, *in a direction parallel to that of the incident ray.* These points are termed the **first and second nodal points** respectively.

The existence of the nodal points of a thick lens with the same medium on both sides of it may easily be inferred.

Let Fig. 75 represent a thick convergent lens, of which A and B are the centres of curvature of the first and second surfaces respectively. From A draw any radius, AP, to the first surface, and from B draw BQ, a radius to the second surface, parallel to AP. Then the small elements of surface at the points P and Q

will obviously be parallel to each other, since they are perpendicular to radii which are parallel. Hence the ray CP, which, after refraction at P, traverses the path PQ, will, on emerging from the lens, follow the path QD parallel to CP; this follows from the fact that the two refractions which it has undergone occurred at opposite, parallel surfaces of a refracting medium. If necessary, produce CP to meet the axis in $N_1$, and produce DQ to meet the axis in $N_2$. Then, $N_1$ and $N_2$ are the first and second nodal points of the lens.

FIG. 75.—Nodal Points of Thick Convergent Lens.

Fig. 76 shows the construction for the nodal points of a thick divergent lens.

There will also be two nodal points for a lens which separates two media of different refractive indices. We shall now determine the position of the nodal points in this case, the most general one possible.

FIG. 76.—Nodal Points of Thick Divergent Lens.

Let $P_1G$, $P_2K$ (Fig. 77) be the first and second principal planes of a lens, and let $F_1$ and $F_2$ be its first and second principal foci. Let $N_1$ be the first nodal point. Let $EN_1$ be any ray directed toward $N_1$. Produce this ray to meet the first principal plane in L. Then, after refraction by the lens, this ray will proceed from M, a point in the second principal plane on the same side of the axis as L, and at a distance $P_2M = P_1L$ from the axis. Also, since $N_1$ is the first nodal point, MQ must be parallel to $EN_1$. Produce QM to meet the axis in $N_2$. Then $N_2$ is the second nodal point.

It at once becomes evident that **the distance between the nodal points $N_1$ and $N_2$ is equal to the distance between the principal points $P_1$ and $P_2$.** For $P_2P_1LM$ and $N_2N_1LM$ are parallelograms with a common side ML, and between the same parallels ML and $P_2N_1$. As a consequence, $P_1N_1 = P_2N_2$; otherwise expressed, the first nodal point is as far in advance of the first principal point as the second nodal point is of the second principal point.

From $F_1$ erect $F_1E$ perpendicular to the axis, and cutting the incident ray $EN_1$ in E. From E draw EG perpendicular to the first principal plane $P_1G$. Then the ray EG, after refraction by the lens, will proceed from K, a point in the second principal plane, on the same side of the axis as G, and at a distance $P_2K$ from it, such that $P_2K = P_1G$; and since EG is parallel to the axis, the refracted ray $KF_2$ will pass through the second principal focus $F_2$.

Further, since E is a point in the first focal plane of the lens, all rays diverging from E will be rendered parallel after refraction by the lens. Hence, $KF_2$ is parallel to MQ, and therefore to $EN_1$.

FIG. 77.—Relation between the Nodal and Principal Points of a Thick Lens.

Thus, the two triangles $KF_2P_2$ and $EN_1F_1$ are equal in all respects, since the corresponding sides of the triangles are parallel, and $KP_2 = GP_1 = EF_1$. Consequently, $F_2P_2 = N_1F_1 = P_1F_1 - P_1N_1$.

Let the distance $P_1F_1$, which is positive and equal to the first focal distance of the lens, be equal to $F_1$; since $P_2F_2$ is equal to the second focal distance of the lens, we may denote it by $F_2$. Then, if $n$ is equal to $P_1N_1$, the distance of the first nodal point in advance of the first principal plane, we have—

$$n = P_1F_1 - F_2P_2 = P_1F_1 + P_2F_2 = F_1 + F_2.$$

When the lens is surrounded by a medium of uniform refractive index, $F_1 = -F_2$ (p. 138). Hence, in this case $n = 0$, and the nodal points coincide with the principal points of the lens. In other cases the position of the nodal points can be determined by the formula just given.

**Use of Nodal Points in determining Magnification.**—Let $N_1$ and $N_2$ (Fig. 78) be the first and second nodal points of a thick lens, and let O and I be an object and the corresponding image. Then, since the ray, from the upper extremity of the object to the first nodal point, is parallel to the ray from the second nodal point to the corresponding extremity of the image, we have—

FIG. 78.—Use of Nodal Points to determine Magnification.

$$m = \frac{I}{O} = \frac{v'}{u'}$$

$$v' = \text{distance } N_2 I,$$
$$u' = \quad ,, \quad N_1 O.$$

We may express this result differently, by stating that the object and image subtend equal angles at the first and second nodal points respectively.

When $v$ and $u'$ have opposite signs, as in the upper drawing (Fig. 78), $m$ is negative. In such a case the image is the inversion of the object. When, as in the lower drawing (Fig. 78), $u'$ and $v'$ have similar signs, the image I is erect.

**Cardinal Points.**—The discovery of the properties of the principal points of a thick lens was originally made by Gauss. Listing investigated the properties of the nodal points. The principal points, nodal points, and principal foci of a lens are termed its **cardinal points.**

The position of the cardinal points of any lens can be determined by the aid of the formulæ already obtained. It can be shown, by the aid of these formulæ, that, in the case of a lens surrounded by a uniform medium, the principal points (which in this case coincide with the nodal points) will be situated within the lens if the latter be biconvex or biconcave. A plano-convex or plano-concave lens will have one principal point in the curved

surface, and the other inside the lens at a numerical distance from the plane surface equal to $\frac{t}{\mu}$. In a concavo-convex lens, the principal points may both be outside the lens (Fig. 76, p. 145).

**Cardinal Points of a System of Lenses.**—Equations (12) and (13), p. 73, for determining the relative positions of image and object with respect to two thin co-axial lenses separated by a finite distance, are of the same general form as equations (1) and (2), pp. 135–6. Thus the solution of equations (12) and (13), p. 73, can be found by making suitable changes in equations (11), (12), (13), pp. 137–8. Thus, we see that a combination of two thin co-axial lenses in air possesses two principal points, which coincide with the nodal points; the combination also possesses two focal distances, equal in magnitude, but opposite in sign, which must be measured from the corresponding principal points. A single *thick* lens, having the same principal points and focal points as the combination, will be exactly equivalent to the latter. The magnification produced by a lens combination can thus be determined by the method explained with relation to a single thick lens.

**Experimental Determination of the Nodal Points of a Thick Lens, or a System of Lenses.**—As we have seen, an incident ray, proceeding toward the first nodal point of a lens or system of lenses, gives rise to a parallel ray proceeding from the second nodal point. Dr. Clay has utilised this law in designing an elegant experimental method of determining the nodal points of a thick lens, or system of lenses. It can be used in connection with a photographic lens system, the focussing lens system of an optical projection lantern, etc.

1. *Convergent Lens, or Lens Combination.*—The lens (or lens combination) is mounted, with its axis horizontal, in front of a white screen provided with an illuminated aperture. On the side of the lens remote from the screen is placed a vertical plane mirror. When the lens is adjusted so that the illuminated aperture is at its first focal point, rays from the aperture are rendered parallel after traversing the lens, and can be reflected back along their original path by the plane mirror; an image of the aperture is then formed on the screen (p. 112). In general, if the lens is rotated about a vertical axis intersecting the optic axis, the image on the screen will be displaced; but if the axis of rotation passes through the first nodal point of the lens, the ray from the centre

of the aperture to the first nodal point will always coincide with the optic axis, and the transmitted ray will always be parallel to the optic axis. In this case, if the transmitted ray is reflected normally from the mirror for any position of the lens, it will still be reflected normally when the lens is rotated through a small angle, and the image on the screen will not be displaced.

The lens is mounted on a carriage which can be rotated about a vertical axis, and the position of the lens with respect to this carriage is adjusted so that the image formed on the screen is not displaced when the carriage is rotated through a small angle. The intersection of the optic axis by the axis of rotation will then give the first nodal point of the lens; the distance from the axis of rotation to the illuminated aperture gives the first focal distance of the lens.

Turning the lens end for end, and repeating the above adjustments, the other nodal point can be found.

The nodal points of a lens may also be found by determining the focal length of the lens by the magnification method described on p. 115, and then measuring this distance from the first principal focus, determined by the method described on p. 112.

2. *Divergent Lens, or Lens Combination.*—In this case a convergent lens is mounted on a separate stand between the lens to be tested and the aperture, the remaining arrangements being the same as described above. The first adjustment is to obtain a distinct image on the screen. The lens to be tested is then adjusted with respect to the carriage on which it is mounted, until its rotation produces no displacement of the image. The intersection of the optic axis by the axis of rotation gives the first nodal point. The lens to be tested is then removed from its carriage, and the distance between the axis of rotation of the latter, and the real image formed by the convergent lens, is measured; this distance is equal to the focal length of the lens to be tested. Turning the lens to be tested end for end, and repeating the above procedure, the other nodal point can be found.

**The Optical System of the Eye.**—The methods developed in the present chapter may be extended to embrace the solution of a problem at once interesting and important, viz. the determination of the cardinal points of the human eye. When these have been found, the most important problems in relation to the refraction of the eye admit of a ready and simple solution.

The structure and functions of the eye will be fully described in Chapter VIII. For the present, we may confine ourselves to an investigation of the refractive properties of the eye. For this purpose

it is necessary to know the positions and curvatures of the various refracting surfaces, and the refractive indices of the various media. These are as follow :—

| | | |
|---|---|---|
| *Radius of curvature of anterior surface of cornea* | $= -$ 7·829 | mm. |
| *Refractive index of cornea and aqueous and vitreous humours* | = 1·3365 | |
| *Crystalline lens :—* | | |
| *Thickness* | = 3·60 | ,, |
| *Radius of curvature of anterior surface* | $= -$ 10·00 | ,, |
| ,, ,, ,, *posterior* ,, | $= +$ 6·00 | ,, |
| *Distance from anterior surface of cornea to anterior surface of lens* | = 3·60 | ,, |
| *Effective refractive index of lens* | = 1·4371 | |

We must now proceed to find the cardinal points of the eye. When these have been obtained, problems involving the determination of ocular images can be solved in the manner already explained.

**Cardinal Points of the Crystalline Lens.**—These will correspond to the cardinal points of a lens of 3·6 mm. thickness, composed of a substance of which the refractive index is equal to 1·4371, and surrounded by a medium of which the refractive index is equal to 1·3365. Thus, $\mu$, the effective refractive index of the lens when surrounded by this medium, will be equal to $\frac{1·4371}{1·3365} = 1·0753$. Then (p. 135),

$$\frac{1}{f_1} = -\frac{\mu - 1}{r_1} = -\frac{·0753}{-10} = ·00753. \quad \therefore f_1 = 132·8 \text{ mm.}$$

$$\frac{1}{f_2} = \frac{1-\mu}{r_2} = -\frac{·0753}{6} = -·01255. \quad \therefore f_2 = -79·7 \text{ mm.}$$

To find the **first and second principal points**, we have (pp. 137–8),

$$a = \frac{f_1 t}{\mu(f_2 - f_1) + t} = \frac{132·8 \times 3·6}{1·075(-79·7 - 132·8) + 3·6} = -2·12 \text{ mm.}$$

$$\beta = \frac{f_2 t}{\mu(f_2 - f_1) + t} = \frac{-79·7 \times 3·6}{1·075(-79·7 - 132·8) + 3·6} = +1·27 \text{ mm.}$$

The **first principal focal length**, $-F$, is given by the equation (p. 137),

$$\therefore F = \frac{\mu f_1 f_2}{\mu(f_2 - f_1) + t} = \frac{·075 \times 132·8 \times (-79·7)}{1·075(-79·7 - 132·8) + 3·6} = 50·61 \text{ mm.}$$

**Cardinal Points of the Eye as a Whole.**—We may neglect the effect of the refraction of the cornea, and treat the anterior surface of the latter as the convex, bounding surface of the aqueous humour, for which the refractive index, $\mu'$, is equal to 1·3365. The radius of curvature ($r_0$) of the anterior surface of the cornea is equal to $-7·829$ mm.; the connection between $u$, the distance of an object, and $v'$, that of the corresponding image, is given by—

$$\frac{\mu'}{v'} - \frac{1}{u} = -\frac{1}{\phi}, \quad \ldots \ldots \quad (a)$$

where $\quad \frac{1}{\phi} = -\frac{\mu' - 1}{r_0} = -\frac{·3365}{-7·829}. \quad \therefore \phi = 23·27 \text{ mm.}$

The first principal point of the crystalline lens is, as proved above, at a distance of 2·12 mm. behind the anterior surface of the lens, and therefore at a distance $3·6 + 2·12 = 5·72$ mm. behind the anterior surface of the cornea. Call this distance $t$. Now we have found a very simple relation between the distances of object and image, when these are respectively measured from the first and second principal points of a lens (p. 138). The image formed by refraction at the cornea will act as object in the refraction by the crystalline lens; its distance from the first principal point of the lens will obviously be equal to $v' + t$. Then, if $v$ is the distance from the second principal point of the lens to the image ultimately formed, we have—

$$\frac{1}{v} - \frac{1}{v' + t} = \frac{1}{F} \quad \ldots \ldots \quad (b)$$

Our object must now be to bring ($a$) and ($b$) into forms similar to those of equations (1) and (2), pp. 135–6. Equation ($a$) needs no modification; we must, however, multiply ($b$) throughout by $\mu'$.

$$\frac{1}{\frac{v}{\mu'}} - \frac{\mu'}{v' + t} = \frac{1}{\frac{F}{\mu'}} \quad \ldots \ldots \quad (c)$$

Equations ($a$) and ($c$) are obtained from equations (1) and (2), pp. 135–6, by leaving $u$ unchanged, altering $\mu$ into $\mu'$, writing $\phi$ for $f_1$, and $\frac{v}{\mu'}$ for $v$, and lastly, writing $\frac{F}{\mu'}$ for $f_2$. To avoid confusion, we also write $\alpha'$ and $\beta'$, instead of $\alpha$ and $\beta$, and $-F$ for the first principal focal distance of the eye. The solution of ($a$) and ($c$) can be obtained by making these changes in equation (4), p. 136, and equations (11), (12),

(13), pp. 137–8; we may write the solution of (*a*) and (*c*) above, in the form—

$$\frac{1}{\frac{v}{\mu'} - \beta'} - \frac{1}{u - \alpha'} = \frac{1}{\mathbf{F}}.$$

$$\therefore \frac{\mu'}{v - \mu'\beta'} - \frac{1}{u - \alpha'} = \frac{1}{\mathbf{F}}, \quad \ldots \ldots \quad (d)$$

where

$$-\mathbf{F} = \frac{\mu'\phi\left(\frac{F}{\mu'}\right)}{\mu'\left(\frac{F}{\mu'} - \phi\right) + t}$$

$$= \frac{23{\cdot}27 \times (-50{\cdot}61)}{-50{\cdot}61 - 1{\cdot}336 \times 23{\cdot}27 + 5{\cdot}72} = 15{\cdot}50 \text{ mm.}$$

The value of $\alpha'$ will determine the position of the **first principal point** of the eye. Since $u$, in equation (*d*), is measured from the anterior surface of the cornea, $\alpha'$ will be measured from the same point. We have—

$$\alpha' = \frac{\phi t}{\mu'\left(-\frac{F}{\mu'} - \phi\right) + t} = \frac{23{\cdot}27 \times 5{\cdot}72}{-50{\cdot}61 - 1{\cdot}336 \times 23{\cdot}27 + 5{\cdot}72}$$

$$= -1{\cdot}75 \text{ mm.}$$

Since $v$ in equation (*d*) is measured from the second principal point of the crystalline lens, it is obvious that $\mu'\beta'$, which determines the position of the **second principal point** of the eye, will be measured from the same point. The value of $\beta'$ is found by making the alterations already specified in equation (12), p. 137.

We have—

$$\mu'\beta' = \frac{\mu'\left(\frac{F}{\mu'}\right)t}{\mu'\left(\frac{F}{\mu'} - \phi\right) + t} = \frac{-50{\cdot}61 \times 5{\cdot}72}{-50{\cdot}61 - 1{\cdot}336 \times 23{\cdot}27 + 5{\cdot}72}$$

$$= +3{\cdot}81 \text{ mm.}$$

To find the distance of the second principal point of the eye from the anterior surface of the cornea, notice that the second principal point of the *lens* is 1·27 mm. in front of the posterior surface of the lens, and the latter is 7·2 mm. behind the anterior surface of the cornea. Hence, the second principal point of the lens is 7·20 − 1·27 = 5·93 mm. behind the anterior surface of the cornea. The second principal point of the eye is, as just found, 3·81 mm. in front of the second principal point of the lens; hence, it will be 5·93 − 3·81 = 2·12 mm. behind the anterior surface of the cornea.

If V and U are the corresponding distances of image and object, respectively measured from the second and first principal points of the eye, then we may write—

$$\frac{\mu'}{V} - \frac{1}{U} = \frac{1}{F}.$$

When $V = \infty$, we have $U = -F$. Hence, $-F$ is the **first principal focal distance** of the eye, measured from the first principal point of the eye. Since $-F = +15{\cdot}50$ mm., we see that the first principal focus of the eye is at a distance of 15·50 mm. in front of the first principal point of the eye, or $15{\cdot}50 - 1{\cdot}75 = 13{\cdot}75$ mm. in front of the anterior surface of the cornea.

When $U = \infty$, we have $V = \mu'F$. This value of V is obviously equal to the **second principal focal distance** of the eye, measured from the second principal point of the eye ; its value is equal to $-1{\cdot}336 \times 15{\cdot}50 = -20{\cdot}71$ mm. Since the second principal point of the eye is at a distance of 2·12 mm. behind the anterior surface of the cornea, the second principal focus of the eye is at a distance of $(20{\cdot}71 + 2{\cdot}12) = 22{\cdot}83$ mm. behind the anterior surface of the cornea.

The distance between the two **Nodal Points** of the eye is equal to the distance between the principal points of the eye (p. 146), *i.e.* $2{\cdot}12 - 1{\cdot}75 = 0{\cdot}37$ mm. Also, the distance of the **first nodal point** in advance of the first principal point of the eye is equal to the algebraical sum of the focal distances of the eye (p. 146). Hence, the first nodal point is at a distance equal to $15{\cdot}50 - 20{\cdot}71 = -5{\cdot}21$ mm. from the first principal point of the eye, or $5{\cdot}21 + 1{\cdot}75 = 6{\cdot}96$ mm. behind the anterior surface of the cornea. The **second nodal point** will be at a distance $6{\cdot}96 + 0{\cdot}37 = 7{\cdot}33$ mm. behind the anterior surface of the cornea.

We may now conveniently collect together the information we have acquired relative to the positions of the **cardinal points of the eye.** All distances are measured from the anterior surface of the cornea, with the ordinary convention as to signs.

| | |
|---|---|
| First Focal Point | + 13·75 mm. |
| Second Focal Point | − 22·83 mm. |
| First Principal Point | − 1·75 mm. |
| Second Principal Point | − 2·12 mm. |
| First Nodal Point | − 6·96 mm. |
| Second Nodal Point | − 7·33 mm. |

**The Schematic Eye.**—The upper diagram, Fig. 79, represents schematically the positions of the cardinal points of the eye, relative to the cornea, crystalline lens, etc. In order that a distant object should be seen clearly when the eye is at rest, the second focal point, $F_2$, must coincide with the intersection of the retina and the optic axis. The first focal point, $F_1$, is the only one of the cardinal points which lies outside the eye. It will be noticed that the distances between the principal points, $P_1$ and $P_2$, and between the nodal points, $N_1$ and $N_2$, are very small.

FIG. 79.—Schematic Eye and "Reduced" Eye.

**The Reduced Eye.**—In order to abbreviate calculations, Listing proposed a simplified optical arrangement which should be equivalent (to a first approximation) to the schematic eye figured above. The principal foci, $F_1$ and $F_2$ (lower diagram, Fig. 79), were to have the same positions as in the schematic eye, but refraction was to occur only at a single curved surface, which we may term the *reduced cornea*. In this case there would be only one principal point, coinciding with the intersection of the refracting surface by the optic axis. This point was taken half-way between the two principal points of the schematic eye. The distance of the first and second principal foci from the reduced cornea would therefore be $+ 15·63$ mm. and $- 20·90$ mm. respectively. The ratio of the focal distances of a refracting surface is equal to $\mu$, the refractive index of the refracting medium; thus the refractive index of the reduced eye must be equal to $\frac{20·90}{15·68} = 1·332$. Let $r$ be the radius of curvature of the refracting surface. Then the

reciprocal of the first focal distance is equal to $\frac{\mu - 1}{r}$. Hence,

$$\frac{\cdot 332}{r} = \frac{1}{15\cdot 68}; \qquad \therefore r = 5\cdot 20 \text{ mm.}$$

There is only one nodal point for a single refracting surface, and this lies at the centre of curvature of the surface. Hence, the single nodal point of the reduced eye lies 5·20 mm. behind the reduced cornea. This agrees almost exactly in position with the mean position of the nodal points in the schematic eye, which lies 7·14 mm. behind the true cornea, and therefore $7\cdot 14 - 1\cdot 93 = 5\cdot 21$ mm. behind the reduced cornea.

Donders has proposed a less accurate, but more useful, reduced eye than the above. He takes the focal distances as 15 mm. and $-20$ mm. respectively, which give a refractive index of $\frac{4}{3}$ (that of water), and for the reduced cornea a radius of curvature of 5 mm. Calculations relative to the size of retinal images, etc., are very much simplified by the use of Donders's reduced eye.

PROBLEM.—In an eye otherwise normal, the distance from the posterior surface of the crystalline lens is 2 mm. longer than usual. Where will be the *punctum remotum* (most distant point which can be seen clearly)?

If V and U are the distances of image and object, measured from the principal points of the eye, of which the first focal distance is $-F$, and $\mu'$ is equal to the refractive index of the vitreous humour, we have (p. 153)—

$$\frac{\mu'}{V} - \frac{1}{U} = \frac{1}{F}.$$

In the normal eye, the second principal focus falls on the retina, at a distance equal to 20·71 mm. behind the second principal point. In the present case the retina is 22·71 mm. behind the second principal points. Therefore $V = -22\cdot 71$. Further, $F = -15\cdot 50$, and $\mu' = 1\cdot 336$. Then—

$$-\frac{1\cdot 336}{22\cdot 71} - \frac{1}{U} = -\frac{1}{15\cdot 5}.$$

$\therefore U = 158$ mm. in front of the first principal point
$= 156$ mm. (nearly) in front of the cornea.

Using Donders's reduced eye, and the formula for refraction at a spherical surface (p. 60), we have—

$$-\frac{\frac{4}{3}}{22} - \frac{1}{u} = -\frac{1}{15}; \quad \therefore u = 165 \text{ mm.}$$

Of course this problem is a very extreme one, since it seldom occurs that the *punctum remotum* is as near to the eye as 160 mm. In ordinary cases, the solution, using Donders's reduced eye, approximates more closely to the true solution.

PROBLEM.—The *punctum remotum* of a particular eye is at a distance of 1 metre from the anterior surface of the cornea. How much longer is the eye than the normal eye?

$$\frac{1·336}{V} - \frac{1}{1001·75} = -\frac{1}{15·50};$$

$$\therefore V = -21·03 \text{ mm.}$$

The normal distance from second principal point to retina is equal to the second focal distance of the normal eye, *i.e.* to − 20·71 mm.

∴ Excess in length of eye in question above normal eye = 21·03 − 20·71 = ·32 mm.

PROBLEM.—In the last problem, if an object 10 mm. in height is placed at the *punctum remotum*, what will be the height of the retinal image?

Using the formula for magnification given on p. 144, we have—

$$\frac{i}{10} = \frac{F_1}{F_1 - U} = \frac{15·5}{15·5 - 1000},$$

$$\therefore i = -10 \times \frac{15·5}{985} = -0·157 \text{ mm.}$$

The negative sign denotes that the image is inverted.

Let us now solve this problem, using Donders's reduced eye. Since object and image subtend equal angles at the single nodal point, which is 5 mm. behind the reduced cornea, and therefore 20 − 5 = 15 mm. from the normal retina, or 15·32 from the retina in question. As this is only an approximate calculation, we may take distance from nodal point to retina in

the eye in question as 15 mm. Similarly, we may take distance of object from nodal point as 1000 mm.

Then, disregarding signs, we have—

$$\frac{i}{15} = \frac{10}{1000}; \quad \therefore i = 0{\cdot}15 \text{ mm.}$$

## Questions on Chapter VII

1. Define (1) the principal points, (2) the principal foci of a lens. Show also that if $f_1, f_2$, are the principal focal lengths of a convex lens, while $u_1$, $u_2$, are the distances, measured from the principal points, of an object and its image, then $\frac{f_1}{u_1} + \frac{f_2}{u_2} = 1$.

2. What do you understand by the nodal and focal points of a system of coaxial lenses? Show how, if the positions of these points are known, the image of any point on the axis, formed by refraction through the system, may be found, and prove that every lens has two nodal points.

3. Prove that if an object be at such a distance from a thick convex lens that it forms an image of equal size at the other side, the distance from object to image, minus the distance between the two principal points (or optical centres) is equal to four times the focal length.

4. Give a careful drawing, as nearly as you can to scale, of the passage of the pencils of rays from an arrow-head, through a sphere of glass whose refracting index is 1·5 and radius 1 inch. The distance of the arrow from the centre of the sphere is 3 inches.

5. Obtain the relation between conjugate foci when a narrow pencil of light is directly refracted at a spherical surface. Prove that a lens formed of a sphere of glass of index $\mu$ and radius $a$ acts like a thin converging lens of focal length $-a/2(1 - \mu^{-1})$, situated at the centre of the sphere.

6. Two thin coaxial lenses are placed at a distance $d$ apart. Calculate the single equivalent lens. Is there any point on the axis of the pair such that if the distances of the object and image are measured from this point the simple formula for the conjugate foci of a single lens may be applied? If so, determine its position.

7. Two similar plano-convex lenses are placed with their plane faces together and then drawn apart to a short distance. Show that when

separated the combination has a greater focal length than when they are in contact. Show also that, when separated, the positions of the principal foci are nearer to the respective curved surfaces than when the lenses were in contact.

8. If $P_1$ and $P_2$ be the respective powers of two thin lenses placed coaxially at a distance $d$ apart, show that the resultant power of the combination is given by the formula—

$$P_1 + P_2 - P_1P_2d.$$

Show also that if the power of a lens or lens combination, capable of giving a real image, be P units in metric measure, and if such a lens is so placed with respect to an object as to give an accurate image on a screen 1 metre distant from the principal plane of the lens, the magnification of the image will be numerically equal to $(P-1)$.

9. Describe the structure of the human eye so far as is necessary for its study in geometrical optics; and state the uses of its various parts. If, in a normal eye, the distance of the second nodal point (second optical centre) from the retina is 15 mm., what will be the area covered on the retina by the image of a circular disc 30 cm. in diameter, the centre of which is 2 metres distant from the eye, in the direct axis of vision, and whose plane is at right angles to this axis?

10. Two lenses, of focal lengths $f_1$ and $f_2$, are placed on a common axis at a distance $d$ apart. Find the focal length of the combination, and the positions of the principal points. Show that, for a single thin lens to be equivalent (in the sense defined on p. 74) to the above combination, it must possess a focal length equal to that of the combination, and must be placed in the first principal plane of the combination.

11. A glass hemisphere of radius $r$, and refractive index $\mu$, is treated as a lens, rays passing through it being limited to those nearly coinciding with the axis. Show that one principal point coincides with the intersection of the convex surface with the axis, while the other principal point is within the lens, at a distance $r/\mu$ from the plane surface. Prove also that the focal length of the lens is equal to $r/(1-\mu)$.

## Practical

1. Determine the position of the two principal points (or optical centres) in the given compound lens.

# CHAPTER VIII

## THE EYE

CONSIDERED in its general aspect, the eye consists of a nearly spherical chamber, provided with a circular window on its anterior side. Light from external objects enters the eye by this window, and real images of these objects are formed on the inside of the posterior surface of this chamber.

**General Structure.**—The external coating of the eye, S (Fig. 80), is termed the **Sclerotic.** It consists of dense fibrous tissue, which is nearly opaque over the posterior five-sixths of the spherical surface of the eye. In front of the eye the sclerotic merges into a transparent meniscus, C, termed the **Cornea.** The curvature of the cornea is greater than that of the general body of the eye.

FIG. 80.—The Eye.

Immediately within the sclerotic is a coating of tissue, Ch, liberally supplied with black pigment cells on its internal surface; this coating is termed the **Choroid.** Close to where the sclerotic merges into the cornea, the choroid merges into a contractile diaphragm, I, with a circular orifice near its centre. This diaphragm, termed the **Iris,** is generally coloured; the central orifice is termed the **Pupil.**

Immediately behind the Iris is a transparent, lens-shaped body, termed the **Crystalline Lens.** It is connected with the walls of the eye by means of an annulus of non-contractile tissue, S.L., termed the **Suspensory Ligament.** Thus the eye

is divided, by the crystalline lens and suspensory ligament, into two chambers. The anterior chamber, between the cornea and lens, is filled with a watery liquid, containing a little common salt in solution. This is termed the **Aqueous Humour.** The posterior chamber is filled with a transparent gelatinous medium, which shows signs of cellular structure when treated by appropriate reagents. This is termed the **Vitreous Humour.** The straight line OX, which may be drawn through the centres of the cornea and crystalline lens, is termed the **Optic Axis** of the eye.

The interior surface of the eye is coated with a nearly transparent membrane, liberally supplied with nerve-fibres and blood-vessels. This membrane, termed the **Retina,** is the part of the eye which is sensitive to light. It ends anteriorly in a ragged circular edge, at about the position where the suspensory ligament originates.

**General Functions.**—The cornea, aqueous humour, crystalline lens, and vitreous humour constitute an optical system, of which the function is to form real images of external objects on the retina. Since the images are real, and no intermediate image is formed, they must be inverted (p. 72). The iris acts as an adjustable stop, to regulate the amount of light entering the eye. The retina, under the action of light, in some unknown manner generates nervous stimuli, which travel to the brain by way of the optic nerve, *o* (Fig. 80), and so produce that form of consciousness which we term sight. The function of the pigmentary layer of the choroid is probably merely to absorb the superfluous light, which would otherwise produce a general illumination of the interior of the eye. Similarly, the insides of telescope tubes and photographic cameras are coated with dull black paint.

We must now examine the more important elements of the eye in greater detail.

**The Cornea** in the living eye is transparent and apparently homogeneous. When treated by suitable reagents, however, it shows traces of cellular structure.

It has been found that the external surface of the cornea does not form part of a spherical surface. Helmholtz considered that it formed part of an ellipsoid of revolution, but Sulzer has shown that its shape is more complicated, and does not agree

with any simple surface. It is more flattened above than below, and more flattened on the nasal than on the temporal side. When we speak, therefore, of the corneal radius of curvature, we mean the radius of curvature of the small cap surrounding the point in which the anterior surface of the cornea is cut by the optic axis of the eye.

The curvature of the cornea has been investigated by the aid of an instrument termed the Ophthalmometer. This instrument was invented by Helmholtz, but has been greatly improved by succeeding investigators.

**The Ophthalmometer.**—The optical methods of measuring the radius of curvature of a convex reflecting surface, which have been described in Chapter V., are not readily applicable to corneal measurements, owing to the smallness of the radius of curvature to be determined. Helmholtz determined the radius of curvature of the cornea, in terms of the magnification produced by reflection of an object of known size in its anterior surface.

EXPT. 34.—Hold a lighted candle in front, and a little to one side, of a person's eye, and observe the erect, diminished image of the flame, produced by reflection at the cornea (Fig. 81). With close attention two other images, which are less distinct, can also be seen; these are formed by reflection at the two surfaces of the crystalline lens.

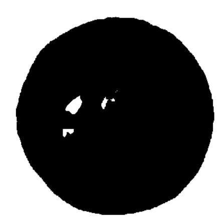


FIG. 81.—Reflected Images in the Eye.

Let $\mathbf{o}$ be the linear dimension of an object placed on the optic axis at a distance $u$ from a reflecting surface; and let $\mathbf{i}$ be the linear dimension of the image produced by reflection at that surface. Then, if $f$ is the focal length of the reflector, we have (p. 38), giving due regard to the signs of the various quantities—

$$m = \frac{\mathbf{i}}{\mathbf{o}} = \frac{-f}{u - f}.$$

In the case of the cornea, $f$ is negative and equal to $-\frac{R}{2}$, where R is the numerical value of the radius of curvature of the surface.

$$\therefore \quad \frac{\mathbf{o}}{\mathbf{i}} = \frac{u + \frac{R}{2}}{\frac{R}{2}}.$$

Further, if $u$ is large in comparison with R, we may write—

$$\frac{o}{i} = \frac{2u}{R}.$$

Since $\frac{o}{i}$ is positive, the image is erect; and since $u$ is greater than R, the image is diminished.

Hence, if we can determine the value of the ratio $o/i$, and if the distance $u$ of the object from the cornea is known, R can be calculated immediately.

The readiest method of measuring the size of a small image is to view it through an optical arrangement which converts it into a double image. This principle was well known prior to Helmholtz, and had been used to measure the diameters of planets. When an image, represented by the central continuous circle in Fig. 82, is viewed through an optical arrangement which produces two images, represented by the broken line circles, it is obvious that if the edges of these latter circles touch each other, as in the figure, then the diameter of the central circle is equal to the distance between the centres of the two broken line circles. Hence, if we determine the distance between the two images of a luminous point, this will be equal to the distance between the centres of the two circular images, and the size of the original image becomes known.

FIG. 82.—Double Image.

The optical arrangement of Javal and Schiötz's ophthalmometer is shown diagrammatically in Fig. 83. This differs only in details from the original arrangement invented by Helmholtz.

The virtual image formed by reflection at the cornea will lie slightly behind the latter. A lens, A, is placed so that the image I is at its principal focus. The light from I, after falling on A, is rendered parallel, and then traverses a Wollaston's prism (Chapter XVIII), P. By this means the parallel pencil is split up into two parallel pencils, equally inclined to the axis. These two pencils fall on a second lens, B, and are focussed at D and E, points in the principal focal plane of B. The lens C is placed so that the real images, D and E, are in its principal focal plane. Thus the rays from D and E are

rendered parallel after traversing C, and the images D and E can be viewed by an eye placed close to the right of C.

FIG. 83.—Ophthalmometer.

The object from which the image I is formed, by reflection in the cornea, consists of two white plates called *mires*, which are supported at $M_1$ and $M_2$, two points equidistant from the axis, by a curved bar, represented by a line in the figure.

If the object viewed were a circular disc in a vertical plane, a diameter of the disc being coincident with the line joining $M_1$ and $M_2$, then, on viewing the image of this formed by reflection at the cornea, through the optical arrangement described above, we should see two circular images; and if we could vary the size of the disc, we could arrange that the images should touch along a horizontal diameter, as in Fig. 84.

In the ophthalmometer as actually constructed, if we represent the mires, $M_1$ and $M_2$, by small arrows pointing outwards, we should see two images, $M_1'$ and $M_1''$, of $M_1$, and two images, $M_2'$ and $M_2''$, of $M_2$. By moving the mires closer together or farther apart, we can make the corresponding extremities of $M_2'$ and $M_1''$ just touch each other. The position of the mires on the supporting bars can then be observed.

FIG. 84.—Double Images in Ophthalmometer.

The supporting bar can be calibrated by observing the positions of the mires when the images are adjusted as described above, the cornea being in turn replaced by various convex spherical surfaces of known curvatures.

The supporting bar carrying the mires, together with the optical

arrangements, including the Wollaston's prism, can be rotated about the axis of the instrument, so that the curvature of the cornea in different planes can be determined. If the mires are adjusted in the horizontal position of the supporting bar, we determine the radius of curvature of the cornea, in the horizontal plane. On rotating the supporting bar, etc., into a vertical plane, the images of the mires will overlap or separate from each other, if the curvature of the cornea is different in different planes.

By the aid of his ophthalmometer, Helmholtz found that, in the neighbourhood of the optic axis of the eye, the radius of curvature of the external surface of the cornea is usually equal to 7·829 mm.

**The Crystalline Lens** is situated with its anterior surface 3·6 mm. behind the anterior surface of the cornea. Of its two surfaces, the posterior one has the greater curvature. The lens itself is not homogeneous, but consists of numerous concentric layers, increasing in density from the outer to the central portion, the whole being enclosed in a transparent capsule. For purposes of calculation, Helmholtz divided the crystalline lens into three portions: an outer or cortical layer; an intermediate layer; and a double convex nucleus. From measurements of the refractive indices of these portions, he calculated the refractive index of a homogeneous lens, of the actual dimensions and focal length of the crystalline lens. He found the refractive index of this *equivalent lens* to be equal to 1·4371.

FIG. 85.—Crystalline Lens.

Reference to Fig. 85 will show that the refractive index of this equivalent lens is not equal to the arithmetical mean of the refractive indices of the various portions of the actual lens. For, if we divide the lens into a nucleus and two menisci, the two surfaces of each meniscus being concentric, then it is clear that the power of the nucleus would be greater than that of the whole lens, even if the latter were homogeneous, for the radii of curvature of the nucleus are smaller than those of the lens, and the focal length of the nucleus is smaller

in the same proportion. The two convexo-concave menisci will act as divergent lenses, since the bounding surfaces of each, being concentric, must be unequally curved; the addition of these to the nucleus will form a combination with greater focal length, and smaller power, than the nucleus; and thus the smaller the refractive indices of the menisci, the greater will be the power of the combination. Thus, the actual crystalline lens will possess a greater power, and a shorter focus, than if the whole lens were homogeneous, and of the same refractive index as the nucleus. In other words, the refractive index of the equivalent lens is greater than that of the nucleus.

The arrangement of the crystalline lens in concentric layers, increasing in refractive index as the centre of the lens is approached, will tend to diminish its spherical aberration. For, in a homogeneous lens, the peripheral portion acts as if it possessed a shorter focus than the portion near the axis. A glance at Fig. 85 will show that the refractive index of peripheral portions of the crystalline lens will be less than that of the portion near the axis; and this will tend to increase the focal length of the peripheral portions, and thus to neutralise the effect of spherical aberration.

The following experiment[1] proves that, when accommodated for near vision, the eye is over-corrected for spherical aberration.

EXPT. 35.—Look, with one eye, at the upper edge of Fig. 85, placed *just beyond* the shortest distance of distinct vision. Now cover the pupil progressively from below, by means of a card; just before the edge of the figure vanishes, it will be seen to sink. The upper edge of the figure lies on the optic axis of the eye; the rays from it, which traverse the *middle* of the pupil, form an image where the retina is cut by the optic axis. Those traversing the *upper edge* of the pupil are insufficiently deviated, and thus form an image above the true one; the mental inversion of ocular images (p. 167), causes the image to appear to sink. Looking at a distant object, and proceeding as before, the image is seen to rise, showing that, when accommodated for distant vision, the eye is under-corrected for spherical aberration.

The eye is also affected by chromatic aberration.

EXPT. 36.—Look through a pin-hole at the line of separation of a roof against a bright sky. Raise the pin-hole, so that the light enters the eye near the periphery of the pupil. The sky just above the roof appears of a reddish colour. If you look at a small flame in the same

[1] "Spherical Aberration of the Eye." By Edwin Edser. *Nature*, April 16, 1903.

manner, the upper portion of the flame will appear blue, and the lower portion red.

Remembering that the image formed on the retina is inverted, and that, when incident near the upper edge of the lens, blue light will be bent down to a greater extent than red light, the result of the above experiment is easily explained.

EXPT. 37.—Look at a distant flame through a piece of glass coloured blue by cobalt oxide. This glass transmits blue, and a certain proportion of red, light. A red image of the flame will be seen, surrounded by a bluish halo.

If a printed page is coloured in alternate vertical stripes of vermilion red and indigo or cobalt blue, it will be found difficult to read the printing, owing to the fact that the focus of the eye must be continually altered in passing from red to blue, and from blue to red.

**Formation of Images.**—Light entering the eye is first refracted at the cornea, of which the refractive index is approximately equal to that of the aqueous humour, *i.e.* to 1·336.

FIG. 86.—Formation of Retinal Image.

It is then refracted at the anterior surface of the crystalline lens, of which the equivalent refractive index is equal to 1·437. A third refraction occurs at the posterior surface of the lens, which is in contact with the vitreous humour, of which the refractive index is equal to that of the aqueous humour. For vision to be distinct, the image of an external object, produced by these refractions, must be formed upon the retina. The formation of a retinal image will be understood by reference to Fig. 86. It will be noticed that the curvature of the image,

which is produced partly by obliquity of the extreme rays, and partly by spherical aberration (pp. 129 and 131), is corrected for, to a greater or less extent, by the spherical shape of the eye.

A detailed investigation of the refraction of the normal eye when at rest has already been given (pp. 149 to 155). It has been proved that the anterior or first focal point of the eye lies at a distance of 13·75 mm. in front of the anterior surface of the cornea. Light radiating from this point will be rendered parallel after traversing the cornea, aqueous humour, and crystalline lens. The posterior or second focal point of the eye lies at a distance of 22·83 mm. behind the anterior surface of the cornea. An axial pencil of rays, parallel before incidence on the eye, will be brought to a focus at the second focal point. Consequently, the image of a distant object will be formed in the second or posterior focal plane of the eye. For a distant object to be clearly seen, it is therefore necessary that the posterior focus of the eye should fall on the retina. An eye which, when at rest, can distinctly see any very distant object, is said to be **emmetropic**; eyes which, when at rest, cannot see a distant object distinctly are said to be **ametropic.**

Since it is necessary, for distinct vision, that a real image should be formed on the retina, it follows that retinal images are inverted. That we see objects erect, is due to the mental interpretation of the retinal images. It is possible to form an erect shadow on the retina, and when this is done, we apparently see an inverted image.

EXPT. 38.—Place a pin-hole at the anterior focus of one of your eyes, and look through this at a bright surface, such as the sky. Hold a pin, head upwards, close to the eye, and an *inverted* shadow of the pin-head will be seen in the pin-hole. Repeat this experiment, using three pin-holes punctured at the corners of a small equilateral triangle, of about 2 mm. side. An inverted shadow of the pin-head will be seen in each of the pin-holes (Fig. 87).

FIG. 87.—Inverted Images of a Pin.

Since the pin-hole is placed at the anterior focus of the eye, light emerging from it will be rendered parallel on entering the eye. Thus, the rays do not cross, as in Fig. 86. As a consequence, an erect shadow of

the pin is thrown on the retina, and this erect image is mentally interpreted as inverted.

When three pin-holes are used, there will be three pencils of light, and each will produce an erect shadow on the retina.

**Accommodation.**—Since the optical system of the eye, when at rest, forms images of distant objects on the retina, it is clear that, unless some change can be effected, near objects will be focussed behind the retina, so that no clear vision of near objects would be possible. Similarly, if a photographic camera is adjusted so that distant objects are sharply focussed on the ground-glass swing-back, near objects will produce blurred images on the ground-glass. In the camera, we must move the swing-back further away from the lens in order to get a sharp image of near objects. In the eye, a change is produced in the optical system by which the image is produced. This change is termed **accommodation.**

EXPT. 39.—In a dark room, hold a candle so that it is in front and toward one side of a person's eye, and look at the images formed by reflection in that eye, from the side opposite to that on which the candle is held. An erect image of the candle is produced by reflection in the cornea. This is very bright, and can be caused to overlap the iris (Fig. 81, p. 161). On the side of this image nearer to the observer will be seen two other images, formed by reflection at the anterior and posterior surfaces of the crystalline lens. The image nearer to the corneal image is erect, and is produced by reflection at the anterior convex surface of the crystalline lens. The remaining image, which is generally very faint, is inverted, and is produced by reflection at the posterior surface of the crystalline lens.

Fig. 81 shows the appearance presented, when the candle is held on the same side of the observed eye as the observer's left hand.

If the person whose eye is observed alternately directs his sight to a distant object, and to an object (such as the finger) held at a distance of about 10 inches from his eye, it will be seen that the act of accommodation produces no appreciable change in the corneal erect image, or the inverted image from the posterior surface of the lens. The image produced by reflection at the anterior surface of the lens is, however, diminished during accommodation for near vision. *This proves*

*that accommodation is effected by an increase in the curvature of the anterior surface of the crystalline lens.*

It will also be noticed that this change is accompanied by a contraction of the pupil, so as to limit the light entering the eye to the more highly refracting portion of the crystalline lens.

We have seen (p. 161) that the magnification, $m$, produced by reflection at a convex reflecting surface, is given by

$$m = \frac{i}{o} = \frac{R}{2u},$$

where R is the numerical value of the radius of curvature of the surface, and $u$ is the distance of the object. If $u$ remains constant, and $m$ diminishes, R must decrease in magnitude.

**The Phakoscope.**—This instrument, invented by Helmholtz, is designed to measure the alteration in the curvature of the anterior surface of the crystalline lens, produced by accommodation for near vision. Two square illuminated discs, one above the other, are reflected in the observed eye, from a position similar to that described in Expt. 34. The images formed by reflection are observed from the side opposite to that on which the discs are placed. The person whose eye is observed alternately directs his vision on a remote object and on a needle placed about 10 inches from his eye. The changes in the reflected images are indicated in Fig. 88. The right-hand figure refers to the unaccommodated eye, while the left-hand figure refers to the accommodated eye.

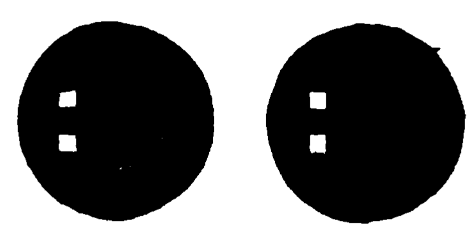


FIG. 88.—Phakoscope Images.

**Mechanism of Accommodation.**—The suspensory ligament, S.L. (Fig. 80), is attached to the anterior capsule layer of the lens. Closely connected with the suspensory ligament is a corrugated ring of involuntary muscular tissue, C.M. (Fig. 80), termed the Ciliary Muscle. Helmholtz considered that when the eye is at rest the suspensory ligament is in a state of tension, so that the anterior capsule layer is tightened and rendered flatter. He further considered that contraction of the ciliary muscle relaxes the tension of the suspensory ligament, so that the anterior surface of the lens can bulge out. Sulzer has, however, shown that the increase in curvature occurs chiefly in the part of the anterior surface of the lens which is nearest to the optic axis. He has produced a similar alteration of curvature, in a lens removed from an eye, by putting the suspensory ligament in a state of tension. According to Sulzer, a contraction of the ciliary muscle puts the suspensory ligament in a state of

tension, and thus tightens the anterior capsule layer, which in its turn squeezes the softer cortical portion of the crystalline lens to one side, so that the anterior surface of lens becomes moulded on the harder and more curved nucleus.

The power of accommodation varies in different individuals, and in the same individual with progressive age. A child of two or three years of age can distinctly see an object placed 2 or 3 inches from the cornea. Adults cannot, as a rule, clearly see an object nearer than 10 or 12 inches from the cornea. With increasing years a further diminution in accommodative power occurs. Aged persons often lose this power almost completely. This is due to a progressive hardening of the cortical layer of the crystalline lens. Loss of accommodative power is termed **Presbyopia.** It is sometimes, mistakenly, termed long-sightedness. A child with normal eyes can see distant objects just as clearly as a presbyopic person, although he can also accommodate his vision for near objects. On the other hand, along with increased hardness, the cortical layer of the lens acquires an increased refractive index, so that the power of the lens is diminished, or its focal length is increased (p. 164), with age. Thus a person who was incapable, in youth, of clearly seeing distant objects, owing to their images being focussed in front of the retina, will lose this defect more or less with advancing years.

The nearest point to the eye at which a small object can be clearly seen is termed the **near point,** or **punctum proximum,** of the eye. The point for which the eye is focussed, when at rest, is termed the **far point,** or **punctum remotum,** of the eye. For normal, or emmetropic eyes, the far point is obviously at an infinite distance (p. 167).

EXPT. 40.—Make two pin-holes about 2 mm. apart, in a sheet of paper. Place these immediately in front of the pupil, and through them view some small bright object (such as the head of a pin), as it is moved to various distances from the eye. When close to the eye, *two* blurred images of the pin's head will be seen. As it is withdrawn, these two images become more distinct, and at the same time draw closer together, till at a certain point a single distinct image is formed. Removing the pin to a greater distance produces no further change. *The nearest position to the eye at which a single distinct image of the pin-head can be seen is the near point of the eye.*

The explanation of the above experiment can be made most obvious by performing another.

EXPT. 41.—By means of a lens, focus the image of a small flame on a white screen, and then cover the side of the lens turned toward the light with a sheet of paper, in which two adjacent holes have been cut. A single image of the flame will still be formed on the screen. Move the screen either toward, or away from, the lens; two blurred spots of light will now be produced.

The explanation of this experiment will be seen at a glance on referring to FIG. 89. All rays from O, a point on the axis,

FIG. 89.—To illustrate Expt. 41.

will be brought to a focus at P, another point on the axis. Hence, a bright object at O will produce an image on a screen placed at P. Covering the front of the lens with a piece of paper in which two holes have been cut, will limit the light traversing the lens to two pencils, both of which, however, converge toward P. If the screen is moved to $P_1$ or $P_2$, these two pencils will cut it at points on opposite sides of the axis, and will produce two small, undefined bright spots.

In Expt. 40, a single distinct image of the pin-head will be seen when its position is such that it is conjugate (p. 33) to a point on the retina, with respect to the accommodated optical system of the eye. When the pin is at a point so close to the eye that its image is formed behind the retina, the conditions are similar to those of Expt. 41, when the screen is at $P_1$ (Fig. 89).

EXPT. 42.—Repeat Expt. 40, using three pin-holes arranged at the angles of an equilateral triangle with apex upwards. When the pin-head is closer to the eye than the near point, *three* blurred images

of the pin-head will be seen, arranged at the angles of an equilateral triangle, *with apex downwards.* Repeat Expt. 41, using, to cover the lens, a sheet of paper with three holes arranged at the angles of an equilateral triangle, with apex upwards. When the screen is placed at $P_1$ (Fig. 89), any point between P and the lens, there will be three undefined bright spots, arranged at the angles of an equilateral triangle *with apex upwards.* In reconciling these results with the eye, remember the mental inversion of retinal images.

**Effects of Cellular Structure.**—Several experimenters have detected traces of cellular structure in the cornea and vitreous humour, when these have been examined microscopically after treatment with suitable reagents. Mr. Shelford Bidwell considers that the results of the following experiment may be explained in terms of the cellular structure of the optical system of the eye.

EXPT. 43.—Look at a brightly illuminated slit (or, better still, the incandescent filament of an electric glow-lamp), through a lens of such power that a properly focussed image is not obtained. A large number of blurred images, lying close and parallel to each other (Fig. 90), will be seen, and not a continuous blurred image, as might have been expected.

FIG. 90.—To illustrate Expt. 43.

EXPT. 44.—By means of a lens, focus the image of an electric lamp filament on a screen. Lay a number of pieces of fine netting (about 1 mm. mesh) face to face, and place them near the lens, and between it and the screen. The image, though of diminished brightness, remains distinct. Now move the screen toward the lens. The blurred image produced resembles that seen by the eye in Expt. 43.

EXPT. 45.—Using the general arrangement of flame, lens, and screen described in Expt. 41, cover that face of the lens which is turned away from the light with a sheet of paper in which three holes are cut. When the screen is at $P_1$ (Fig. 89), three blurred spots of light will be seen.

In Expt. 44 the small apertures in the bundle of netting would produce numerous blurred images of a single luminous point, when the screen and point are not conjugate. Hence,

under the same conditions, each point of the incandescent filament produces a number of blurred points of light on the screen, and the whole filament produces a number of blurred and partially superimposed images.

The lens used in Expt. 43 only serves the purpose of throwing the eye out of focus. According to Mr. Bidwell, the cellular structure of the eye produces an effect similar to that due to the bundle of netting in Expt. 44.

**The Retina** is a transparent membrane, lining the posterior five-sixths of the interior surface of the eye. Its structure will be understood on reference to Fig. 91. The surface in contact with the vitreous humour (which we shall term the internal surface) consists of a very thin layer of connective tissue, J; this and a second (external) layer of connective tissue, E, are bound together by transverse bundles of connective tissue, C.T., the intermediate spaces being mostly occupied by nerve tissue. The optic nerve, *o* (Fig. 80), enters the eye on the nasal side of that point of the retina which is cut by the optic axis, and gives rise to nerve filaments, N (Fig. 91), most of which are destitute of the usual medullary sheath. These spread out through the layers immediately beneath the internal layer of connective tissue, and end in ganglion cells, G, which send processes into a finely reticulated layer, Q, of nerve tissue. Thence, filaments characterised by nucleated swellings penetrate the retina transversely, in the manner shown in Fig. 91, till, on reaching the external layer of connective tissue, E, they become continuous with a number of small elongated bodies, R, R, and C, C, which are packed closely together, side by side, and form a layer termed the **Bacillary Layer**, or **Jacob's Membrane**. The bodies R, R, are nearly cylindrical, and their extremities are surrounded with pigment cells (not shown in Fig. 91); they are termed **Rods**. The bodies C, C, are shorter, and are shaped somewhat like flasks; they are termed **Cones**.

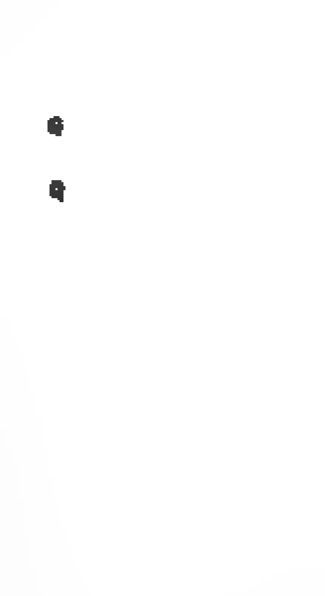


FIG. 91.—Transverse Section of Human Retina.

**Purkinje's Figures.**—An artery enters the eye along the axis of the optic nerve, and ramifies in the internal layer of the retina, immediately beneath the layer of connective tissue, J (Fig. 91 ; see also Fig. 80). It is possible for the eye to see its own blood-vessels. The visualisation, by an eye, of its own blood-vessels gives rise to the phenomena termed Purkinje's Figures.

EXPT. 46.—By means of a lens of 4 or 5 inches focus, form an image of a lamp or gas flame on the sclerotic close to its junction with the cornea, where it is thin and light can penetrate it. If the eye, illumined in this manner, is directed toward a dark surface, a black, tree-like image, on a luminous, slightly rosy, background, will be seen. If the bright image on the sclerotic is moved, the tree-like image will also move. It is the shadow of the retinal blood-vessels, thrown on the sensitive layer of the retina.

In Fig. 92, let an image of the flame be formed near the limiting edge of the sclerotic. Some light penetrates the sclerotic and illumines the neighbouring choroid, and light radiates from this point, and falls on the retina in the neighbourhood of D. Let C be the section of a retinal blood-vessel. Then the shadow of this will be cast on the sensitive layer of the retina at D. To the eye this shadow will appear as if it were the image of an external object, situated somewhere in the line DNF, which passes from D through the mean nodal point N. (For a more accurate construction, see p. 147.) If the image of the flame on the sclerotic is moved away from the cornea, the shadow of C on the sensitive layer of the retina will move to E, and this will appear as the image of an external object situated somewhere in the line ENG. Thus, as the image of the flame moves round the eye, the visually projected image of the blood-vessels will move in the same direction.

FIG. 92.—To illustrate Expt. 46.

This proves that the sensitive layer of the retina is not imme-

diately beneath the blood-vessels. By careful observation of the angle FNG, corresponding to a displacement of the image of the flame through a measured distance on the sclerotic, the exact position of the sensitive layer of the retina can be localised.

The angle FNG, together with the distance NE, which is known, determine the distance DE through which the shadow of the blood-vessel, C, actually travels. We can then easily calculate how far C must be in advance of the sensitive layer of the retina, in order that the motion of a luminous point through a measured distance should move the shadow of C from D to E.

*By this means it has been found that the sensitive layer of the Retina is the Bacillary Layer, or Jacob's Membrane.*

Thus the rods and cones appear to be the ultimate organs of sight.

It may appear strange that we do not always see the shadows of the retinal blood-vessels when we gaze on a bright surface, such as a white cloud. It should be remembered, however, that in ordinary sight light converges to any particular point of the retina from all points of the pupil, and thus the shadow formed would be rendered indistinct. The following experiment will give a clue to the reason why no shadow at all is usually seen.

EXPT. 47.—Place a pin-hole at the anterior focus of the eye, and look through this at a luminous surface, such as a bright sky, or better, a uniform opal globe surrounding a gas flame. Move the pin-hole regularly up and down over the extent of the pupil, at the rate of about one complete to-and-fro motion per second. The *horizontal* branches of the retinal blood-vessels will be distinctly seen, as black shadows on the luminous surface. On stopping the motion, the blood-vessels disappear. If the pin-hole is moved from side to side, the *vertical* branches of the blood-vessels become visible. If the pin-hole is moved, at the same rate as before, in a small circle, the blood-vessels become distinctly visible throughout the whole of their course. Notice that they extend from the outside toward the centre of the field of view, breaking up into smaller branches as the centre is approached, but leaving a small clear space surrounding the centre of the field.

As the pin-hole is moved, the visual projection of the blood-

vessels will be observed to move also. Hence we see that the conditions under which the blood-vessels may be visualised are :—

(1) The light must be rendered parallel (or nearly so) within the eye, so as to throw sharp shadows. This condition is secured by placing the pin-hole at or near the anterior focus.

(2) The shadow must be kept constantly moving from one part of the bacillary layer to another. This is secured by moving the pin-hole in a small circle.

When the shadow of a blood-vessel falls on any part of the bacillary layer, it is at first distinctly seen; but the rods and cones exposed to the full light become fatigued, or less sensitive to light, while those in the shadow suffer less in this respect. Thus, after a short time, the smaller amount of light in the shadow is compensated for by the greater sensitiveness of the rods and cones there, so that a mental impression of uniform illumination is produced.

**Retinal Fatigue.**—The following experiment shows that, after any part of the bacillary layer of the retina has been exposed to light for some time, it becomes less sensitive to light.

EXPT. 48.—Gaze for a short time at a bright object, and then turn your eyes on to a uniform illuminated surface. The shape of the bright object will be seen, projected on to the surface, as a dark patch.

This experiment succeeds best with adults or old people. In youth the bacillary layer recovers its normal state so quickly that it is difficult to detect a fatigue image.

**Persistence of Impressions.**—When the bacillary layer has been excited, it does not cease to generate a sensation of light immediately on the removal of the stimulus. The time required for the sensation to subside is from an eighth of a second (with light of moderate intensity) to a tenth of a second (with bright light). Thus, if the glowing end of a stick is caused to rapidly revolve in a circle, a continuous bright circle will be seen. Rapid fluctuations of brilliancy occur in electric lamps worked by means of alternating currents, yet a stationary object

will appear to be uniformly illuminated. If, however, the object is moved rapidly to and fro, a number of isolated images will be seen.

**Stroboscopic Observations.—**

EXPT. 49.—Mount a small disc of white paper on one end of a straightened piece of watch or clock spring about 4 or 5 inches in length, and clamp the other end of the spring in a vice. Set the spring vibrating; the moving paper disc will be seen as a white oblong, owing to the persistence of visual impressions. Now clamp another similar piece of spring in a vice, at a distance of 10 or 12 inches from the first one, and on its free end mount a larger disc of paper in which a vertical slot is cut, the plane of the paper being parallel to that in which the spring can vibrate. Set both springs vibrating. If the time of vibration is the same in both springs, the small disc on the first one will appear stationary when viewed through the slot in the disc mounted on the second. In this case the small disc can only be seen when the slot in the larger disc is in front of the pupil, and the time of vibration of both springs being equal, the small disc will always be in the same position when light from it reaches the eye. If, as generally happens, the times of vibration of the two springs are not quite equal, the small disc will be seen to move *slowly* to and fro. In this case, each time the slot comes in front of the pupil, the small disc will occupy a position slightly behind, or in advance of, that in which it was last seen, according as the small disc moves to and fro in a greater or less time than the slotted disc. The time of vibration of either spring can be varied by loading its free end with shot or small pieces of lead, attached by means of a small quantity of soft wax.

The Stroboscopic method of viewing a moving object is often employed in physical investigations, as it gives us the opportunity of studying in detail the nature of the motion.

**Irradiation.**—The white square on a black ground, in Fig. 93, appears to be larger than the neighbouring black square on a white ground, although measurement will prove that both are exactly equal in dimensions. This phenomenon, termed irradiation, is probably due to the excitation of the rods and cones adjacent to, but not absolutely within, the geometrical image of a bright object, by means of light reflected from the tissue of the retina within the image. Irradiation is particularly noticeable when the moon, in her first quarter, is seen to consist

of a bright meniscus and a more dimly lighted portion ("the old moon in the new moon's arms"). The meniscus appears to

FIG. 93.—Irradiation.

belong to a circle of greater diameter than that of the more dimly lighted portion.

**The Yellow Spot and Fovea Centralis.**—At a small distance toward the temporal side of the point of the retina which is cut by the optic axis, there is a small pit in the retina, F (Fig. 80), called the **Fovea Centralis.** The immediately surrounding portion of the retina is of a yellowish colour, and is destitute of blood-vessels, except the finest capillaries. This portion of the retina is termed the **Yellow Spot,** or **Macula Lutea.** The yellow spot is more sensitive to light than the rest of the retina. When we look directly at a small object, it is focussed on the fovea. Hence, it is obvious that the **visual line,** FV (Fig. 80), is inclined to the optic axis of the eye. The visual line really consists of two straight lines, one from the fovea to the posterior or second nodal point, and another, parallel to the first, from the first or anterior nodal point to the object (p. 147). The two nodal points are, however, very close together, and their mean position coincides very closely with the intersection of the posterior surface of the crystalline lens by the optic axis (Fig. 79).

Thus the central portion of the field of view is focussed on the yellow spot. The absence of blood-vessels from this region was noticed in connection with Expt. 47.

EXPT. 50.—Obtain a glass cell with parallel plane glass sides, and fill this with a solution of chrome alum. Close your eyes for a few minutes, and then look through the cell at a white cloud. For a short time a rosy patch will be seen in the centre of the purple field of vision. This is the visual projection of the yellow spot. The rosy patch quickly disappears, but can be again seen after closing your eyes for some time.

The bichromate solution only allows bluish-green rays, with a certain proportion of red rays, to traverse it, the resulting light being of a purple colour. A portion of the transmitted bluish-green light is absorbed by the pigment in the yellow spot, so that the light falling on the bacillary layer of the latter will possess a rosy tinge in comparison with that traversing other portions of the retina, where no absorption takes place. The disappearance of the rosy patch is due to fatigue of the rods and cones, which will be greatest where the light is strongest, as explained in connection with Expt. 47.

The bacillary layer of the Fovea Centralis is entirely composed of cones, which are longer, more slender, and more closely packed there than in other parts of the retina. (A foveal cone is about 0·002 mm. in diameter; other retinal cones are about 0·006 mm. in diameter.) In order that two small neighbouring points of light should be distinguished from each other, each must fall on a separate cone.

Assuming the foveal cones to be in contact, we can calculate the angle which a small object must subtend at the first nodal point of the eye, in order to be distinctly seen. For object and image respectively subtend equal angles at the first and second nodal points of the eye (p. 147). Also, the distance of the second nodal point from the retina of an emmetropic eye is equal to $(22{\cdot}83 - 7{\cdot}33) = 15{\cdot}5$ mm. (p. 153). Thus, the distance between the centres of two contiguous foveal cones will subtend, at the second nodal point, an angle of $\frac{{\cdot}002}{15{\cdot}5} = 0{\cdot}00013$ radians, or about 26″ of arc. Hence, two small objects a centimetre apart could (theoretically) just be distinguished when placed at a distance of 77 metres from the eye.

**Distinctive Functions of Rods and Cones.**—If a solid is heated to a sufficiently high temperature, it is well known that it emits white light. The incandescent filament of an electric glow-lamp, or the crater of an arc-lamp, may be cited as instances. If the temperature is allowed to fall, ordinary

observation shows that the light becomes more and more reddish in hue, till at last even the dull red glow vanishes. Thus, as a heated body is cooled, light of lower and lower refrangibility is emitted. Even after the last trace of visible radiations have ceased to be emitted, other rays, which we may term infra-red rays, are given off, and these may be detected and examined by means of the bolometer or radiomicrometer.[1]

If in a perfectly dark room a piece of platinum is slowly heated by an electric current, and the first trace of luminosity is watched for by an eye accustomed to the dark, a faint gray glow will be the first thing seen. This "gray glow," as it is termed, has a peculiar flickering appearance, due to the fact that it disappears when looked at directly, but reappears when the eye is turned to a point a little on one side of it. Thus, it is seen that the fovea centralis is insensitive to the gray glow, while surrounding parts of the retina can be affected by it. It will be remembered that the rods are entirely absent from the fovea, while they are plentifully scattered through the rest of the bacillary layer. This has given rise to the theory that the gray glow is perceived by the aid of the rods, but not the cones of the retina.

The gray glow may be due to traces of ordinary light too faint to act on the cones, or may possibly be produced by rays of less refrangibility than the red. The former supposition is perhaps the more reasonable. In the eyes of animals that seek their food in the dark, or dusk (such as the owl and bat), the bacillary layer is entirely composed of rods. Further, faint stars (such as the Pleiades) may be seen more distinctly when the eye is directed a little to one side of them. Sometimes a very faint star, which can be seen when the eye is directed to a neighbouring part of the heavens, will entirely disappear when looked at directly. A piece of paper, illuminated by moonlight which has passed through red glass, will not appear coloured, but of a grayish hue, so that the shadow of a stained glass window, thrown by moonlight on the stone floor of a church, presents merely variations of grayish light.

It would thus appear that the function of the rods in the bacillary layer is to produce consciousness of very faint light, irrespective of colour; while colour sensations are produced by the cones.

**The Visual Purple.**—In man and many animals the terminal, cylindrical portion of the rods (Fig. 91) is of a deep purple

[1] See *Heat for Advanced Students*, by the Author, pp. 404 and 410.

colour. The colouring-matter of these rods, termed *visual purple*, may be dissolved out by appropriate chemical reagents, and a deep purple solution, which is bleached by light, is obtained. Yellowish-green light has the strongest bleaching action. It happens that the yellowish-green part of the spectrum is that which appears brightest to the living eye. Light also produces a bleaching action on the purple colouring-matter in the rods during ordinary vision; the colour becomes gradually restored in darkness. If the eye of an animal is focussed on a bright object immediately before it is killed, a bleached image of the object will be found on the retina, if the eye is not exposed to light during dissection. This image can be "fixed" by washing in a 10 per cent. solution of potash alum, and an ocular photograph of the object thus obtained.

It is uncertain whether the visual purple plays any important part in vision. The rods in the eye of the owl are of a very deep purple colour, while those of the bat are colourless. Since both of these animals seek their food in the dusk, and must have eyes extremely sensitive to faint light, it would appear that the presence of visual purple is not necessary for, although it may exert some unknown influence on, the functioning of the rods.

**The Blind Spot.**—The optic nerve enters the eye on the nasal side of the fovea, where it forms a small eminence which is left uncovered by both the choroid and the retina. We shall therefore be prepared to find that this part of the eye is insensitive to light, unless, indeed, the nerve substance were affected by light, which we have seen reason to believe is not the case.

EXPT. 51.—Close the left eye, and with the right one look directly at the star in Fig. 94. Move the book to and from the eye. When at a distance of about 15 inches from the eye, the circular white spot will disappear. At less or greater distances, the circular spot will be visible. Care must be taken to keep the eye steadily directed toward the star during this experiment.

Thus, there is a spot on the internal surface of the eye which is insensitive to light. The position of this spot can be determined. For we know that the image of the star will be formed on the fovea, and the distance between the star and

white spot in Fig. 94 will subtend, at the anterior nodal point of the eye, an angle equal to that subtended at the posterior nodal point, by the distance from the fovea to the blind spot.

FIG. 94.—To determine the Blind Spot.

Thus, the position of the blind spot has been found to agree exactly with the eminence formed by the entrance of the optic nerve.

**Objective Inspection of the Interior of the Eye.**—A method by which a person can see the blood-vessels of his own eye has already been described (p. 175). This class of observation is termed subjective. A method of viewing the interior of another person's eyes also exists. Such observations are termed objective.

As we have seen, parallel light, on entering an unaccommodated emmetropic eye, is brought to a focus on the retina. A certain amount of red light is reflected from the illumined retina, chiefly by the transverse bundles of connective tissue (Fig. 91) and the blood-vessels. This light would be rendered parallel on leaving the eye, and the internal surface of the retina (termed the *fundus* of the eye) could be seen by another emmetropic eye, were it not that the head of the observer must be placed in front of the observed eye (owing to the smallness of the pupil), and would thus cut off the light which is required to illumine the fundus.

A red glow is often, however, seen in the eye of the horse and dog. This is the light, reflected from the fundus, which reaches the observer's eye, owing to the large pupil of the animal. The green glow of a cat's eye in a dimly-lighted room, when the cat's pupil is distended,

has a similar origin; the peculiar colour is due to cells, said to contain crystalline bodies, which are distributed through the retina of the cat.

The first attempt to view the fundus of the living eye was made by Brücke, who looked into the eye through a tube passing through the flame by which the fundus was illumined. It was by Helmholtz, however, that the problem was finally solved. He used a real image, formed by reflection at a concave mirror, as the source of light, and viewed the fundus through a small central aperture in the mirror. An arrangement designed to view the fundus of the eye in this manner, is termed an Ophthalmoscope.

**The Ophthalmoscope.**—One form which this instrument may take is shown in Fig. 95.

A source of light, *l*, is placed as near as possible to, but a little behind, the eye to be observed. Light from this source falls on a concave mirror, M, provided with a central perforation, A. This mirror is inclined so that the resulting real image, I, is formed in the straight line joining the perforation, A, and the pupil. A lens, L, is placed in front of the eye, in such a position that the image, I, is at its principal focus, so that light from I will be rendered parallel after traversing it, and will be focussed on the fundus of the emmetropic eye, E, without

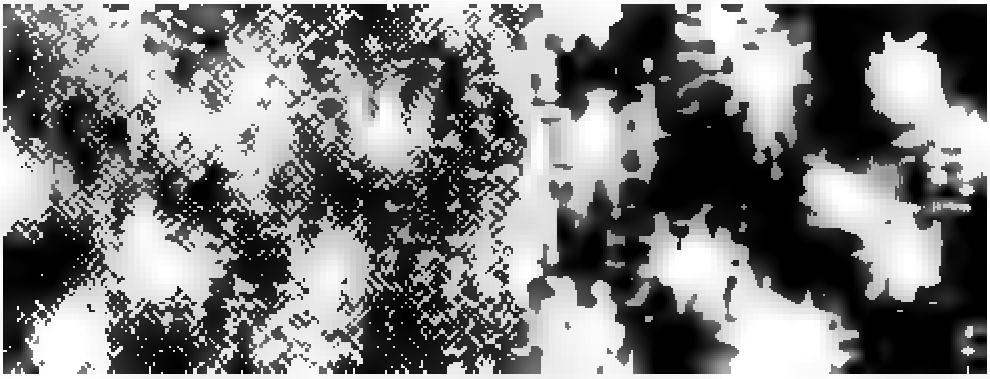


FIG. 95.—Ophthalmoscope.

accommodation. Thus, illumination of the fundus, the first thing necessary for its inspection, is secured. Light from the illuminated fundus will be rendered parallel on leaving the eye, and will be brought to a focus at I. Thus, if the distance from A to I is equal to, or greater than, 10 inches, an eye with normal accommodative power, looking through A, will see a magnified inverted image of the fundus situated at I.

By means of the ophthalmoscope the various parts of the fundus of the human eye can be minutely examined. The yellow spot, fovea centralis, blind spot, and the various nerve-fibres and blood-vessels can all be rendered distinctly visible. By its means a narrow pencil of light can be directed on to the blind spot, and the conclusions reached as a result of Expt. 51 verified. The ophthalmoscope is also valuable in examining the refractive properties of defective eyes.

**Binocular Vision.**—When we view a small object by means of both eyes, an image of the object is formed on the fovea of each eye. Hence, it becomes apparent that, in binocular vision of a near object, the visual lines converge toward that object. This is a matter of some importance, for we learn, by experience, to judge of the distance of near objects in terms of the muscular effort required to produce this convergence.

Expt. 52.—Try to thread a needle with one eye closed. It will be found much more difficult than when both eyes are used.

When we view a solid body with both eyes, two slightly different ocular images are formed. This becomes apparent if we hold one hand edgeways some distance in front of the face, and after looking at it with both eyes, close first one eye and then the other. This difference in the ocular images of a near object we have learnt, by experience, to associate with solidity. Thus, a portrait or photograph, however striking may be its likeness to a person, always has a suspicion of *flatness* in comparison with the actual appearance of the person.

**The Stereoscope.**—In this instrument two photographs of an object, or group of objects, are simultaneously viewed by the two eyes, through lenses which cause the visual lines to converge toward

Fig. 96.—Stereoscopic Vision.

a point, images of corresponding points in the two photographs being formed on the foveas of the two eyes. The two photographs are not exactly similar, but are obtained by the use of two cameras, of which the optic axes converge about as much as the visual lines would when viewing the object to be photographed. Thus, in the stereoscope we see two different views of the object, with an ocular convergence similar to that necessary for viewing the actual object. As a result, we obtain a wonderful appearance of relief in the objects, and depth in the picture. The convergence of the visual lines is generally produced by looking through two de-centred lenses (Fig. 96).

**Defective Eyes.**—As already explained, the emmetropic eye brings parallel light to a focus on the retina. The most common ocular defects arise from the retina being either behind, or in front of, the posterior (or second) focal point of the eye.

MYOPIA.—In Myopia, parallel light is brought to a focus *in front of the retina*, due in general to an excessive length of the eye. As a consequence, such an eye obtains only a blurred image of distant objects. The term myopia was suggested by the practice of nearly closing the eyes when viewing distant objects, which is characteristic of persons suffering from this defect. The light proceeding from each point of a distant object produces, after passing through the pupil, a cone of rays converging to a point in front of the retina, and then diverging, and thus producing a blurred spot on the retina. By diminishing the aperture through which the rays enter the eye, the angle of the cone is diminished, and a smaller spot is produced on the retina. The myopic eye can see near objects distinctly, and its near point is closer to the eye than in the emmetropic eye.

HYPERMETROPIA.— In Hypermetropia parallel light is brought to a focus *behind the retina*, generally due to deficient length of the eye. Such an eye can obtain distinct vision of neither distant nor near objects, unless by an act of accommodation. In the unaccommodated hypermetropic eye, the only light which could be focussed on the retina would be that which converges toward a point behind the retina.

ASTIGMATISM.—The Astigmatic eye has different refractive powers in different planes, often due to irregularity in the

curvature of the cornea. Such an eye may be able to see, for instance, the horizontal twigs of a tree, while the vertical twigs are indistinct or invisible.

APHAKIA.—In Aphakia the crystalline lens has been removed from the eye. In this case, the optical system of the eye is extremely simple, consisting of a single convex surface—the cornea—bounding a medium of refractive index equal to 1·337. The aphakic eye is, of course, incapable of accommodation. An eye which was emmetropic before the extraction of the lens will be hypermetropic after that operation. Extraction of the lens is sometimes resorted to as a cure for excessive myopia.

In the normal eye, the radius of curvature of the cornea is equal to $-7·83$ mm. Hence, the equation connecting $v$ and $u$, the respective distances of image and object, measured from the anterior surface of the cornea, takes the form

$$\frac{1·337}{v} - \frac{1}{u} = \frac{·337}{-7·83} = -\frac{1}{23·2}.$$

To find the position of the posterior focus, put $u = \infty$. Then,

$$v = -23·2 \times 1·337 = -31·0 \text{ mm.}$$

Since the normal distance from the anterior surface of the cornea to the retina is equal to 22·8 mm., it becomes apparent that the second focal point of the aphakic eye is $(31·0 - 22·8) = 8·2$ mm. behind the retina.

To find the position of the first focal point, put $v = \infty$. Then $u = +23·2$ mm. Thus, the first focal point of the aphakic eye is in advance of the position it would occupy in the emmetropic eye.

## QUESTIONS ON CHAPTER VIII

1. When the eye is immersed in water, near objects cannot be distinctly seen. Why is this?

2. If F is the first focal distance of the eye, and $f$ is the focal length of the ophthalmoscope lens L (Fig. 95, p. 183), prove that the image of the fundus is magnified $f/F$ diameters.

# CHAPTER IX

## VISION THROUGH A LENS

**Spectacles.**—In order to remedy the ocular defects described in Chapter VIII, spectacle lenses of various kinds are used. The nature of the lens to be employed to remedy any particular defect can easily be determined.

1. MYOPIC EYE.—In this case near vision is equal, or superior, to that of an emmetropic eye. Since, however, parallel rays are brought to a focus in front of the retina, distant objects cannot be seen distinctly, and the far point, instead of being at infinity, will be at a limited distance in front of the eye. Thus spectacles become necessary for distant vision.

Fig. 97, A, represents the refractive action of a myopic eye. Parallel rays are brought to a focus at F, in front of the retina. Accommodation only serves to reduce the anterior and posterior focal lengths of the eye, so that distinct vision of distant objects cannot be obtained by its aid. Let P be the far point of this eye. Then light diverging from P will be brought to a focus on the retina, without accommodation. In order that distant objects should be seen, it is necessary to employ a divergent lens, L, such that parallel rays, after passing through it, shall diverge from a virtual focus at P. It at once becomes obvious that the distance LP must be equal to the focal length of the lens used. Also, since P is a fixed point with respect to the eye, the distance LP will diminish as the lens is removed from the eye. Thus, the farther the lens is worn from the eye, the greater is the power (p. 74), or the smaller is the focal length, of the lens required to see distant objects. On the other hand, a lens which, when placed at L, gives distinct vision of distant objects without accommodation, can be used to give distinct vision of nearer objects without accommodation by moving it farther from the eye.

2. Hypermetropic Eye.—In this case parallel rays are focussed behind the retina by the eye when at rest. If the accommodative power of the eye is sufficient, distant objects, or even those relatively close to the eye, can be seen by its aid. A great strain is, however, generally imposed on the ciliary muscle, so that it is generally advisable to use suitable spectacles.

Fig. 97, B, represents the refractive action of a hypermetropic eye. Parallel rays are focussed at F, a point behind the retina. In

Fig. 97.—Spectacle Lenses used in conjunction with Myopic and Hypermetropic Eyes.

order to be focussed, without accommodation, on the retina, light must converge toward a point, P, behind the eye. P is thus the far point of the eye (p. 170). When the position of P has been determined, the exact character of the lens required for distant vision becomes known. For the required lens must cause parallel rays, after passing through it, to converge toward P. Thus LP is the focal length of the lens required. As the distance from L to P is measured in the negative direction, a lens of negative focal length (or a converging lens) must be used. Then, if the accommodative power of the eye is normal, objects at any distance down to about 10 inches from the eye can be seen by its aid. Since the point P is fixed with respect to the eye, the distance LP will increase in numerical value as the lens is removed from the eye. Hence, in order to view distant objects, weaker glasses can be used if these are supported at a greater distance from the eye.

The strength of the glass required to view near objects, without ocular accommodation, can be easily calculated. Let $u$ be the distance from the lens at which the object to be viewed is placed. Let the distance LP be equal to $-p$, and let $f$ be the focal length of the lens required. Then—

$$-\frac{1}{p} - \frac{1}{u} = \frac{1}{f}, \qquad \therefore \; -f = \frac{up}{u+p}.$$

The effect of changing the distance from the eye at which the lens is placed, on the focal length of the lens required, can be easily investigated. Let us suppose that the position of the lens is advanced by a distance $d$, small in comparison with $u$ or $p$. Then, to determine $f_1$, the focal length of the lens required, we shall have the equation—

$$\frac{1}{p+d} + \frac{1}{u-d} = -\frac{1}{f_1}$$

$$\therefore \; \frac{u-d+p+d}{up+(u-p)d-d^2} = -\frac{1}{f_1}.$$

Since $d$ is small, we may neglect $d^2$ in comparison with the remaining terms in the denominator. Then—

$$-f_1 = \frac{up+(u-p)d}{u+p}.$$

Both $u$ and $p$ are positive. When $u>p$, that is, when the lens was originally nearer to the far point P than to the object viewed, $(u-p)d$ will be positive, and the numerical value of $f_1$ will be greater than that of $f$. When $p>u$, $(u-p)d$ will be negative, and $f_1$ will be less than $f$. *Thus, if the position of the glass can be changed so as to bring it nearer to the midway point between* P *and the object viewed, a lens of greater focal length, or smaller power, can be used.*

With given spectacles a nearer object may be viewed without accommodation, or the power of the glasses can virtually be increased, by moving them farther from the eye, provided the original distance from lens to object was numerically greater than the distance from the lens to P. When the lens is midway between the object and P, the distance from lens to object will be equal to twice the focal length of the lens, as can be seen by substituting $p=u$ in the above equation for $f$.

3. PRESBYOPIC EYE.—In this case distant objects can be clearly seen, if loss of accommodative power is the only existing defect (p. 170). In order that near objects should be seen, light from them must be rendered parallel before reaching the eye. Hence,

in order that such an eye shall be able to read ordinary type placed at a distance of 12 inches (30 cm.) from the position where the spectacles are to be worn, convex glasses, of $-30$ cm. focal length, or $+3 \cdot 3$ dioptres in power (p. 74), must be used. For distinct vision, a near object must be placed at the focus of the lens used; thus, if the latter is moved farther from the eye, the object must be moved in the same direction to an equal extent.

4. APHAKIC EYE.—This eye is generally hypermetropic, the posterior focus being at a distance of 31·1 mm. behind the cornea (p. 186). It will also be incapable of accommodation. Different convex glasses will generally be required for distant, and for near vision. The strength of the glasses required in either case can be calculated on the same principles as those explained in connection with hypermetropia.

5. ASTIGMATIC EYE.—In this case the refractive power of the eye is different in different planes. This defect is corrected by wearing cylindrical glasses, so as to reinforce the power of the eye in the plane of least curvature of the cornea, or to neutralise the greater power of the eye in the plane of greatest curvature of the cornea. An astigmatic eye may also possess any of the defects previously dealt with, in which case a lens, cylindrical on one surface, and spherical (concave or convex, as the case requires) on the other, becomes necessary.

**Magnification produced by Spectacles.**—When a near object is viewed by the aid of spectacles, a virtual image of it is seen, and the dimensions of the image can be calculated from those of the object, the distance between object and lens, and the focal length of the lens, in the manner explained on p. 72. In this section, however, we must turn our attention to the actual size of the image produced in the eye itself, when an object is viewed through a lens. In what follows, the eye must be supposed to be unaccommodated, unless the reverse is expressly stated. We shall find that with an object at a constant distance from the eye, the size of the retinal image is affected by the position of the lens.

1. EYE WITHOUT LENS.—Having given the principal planes, *ad* and *be*, together with the first and second foci ($F_1$ and $F_2$) of the eye, we may readily construct the ocular image of a small object, O, standing

on the axis (Fig. 98). From the upper extremity of O draw a straight line through $F_1$, cutting the first principal plane at *a*; and from *b*, a point in the second principal plane, on the same side of the axis as *a*, and at an equal distance from it, draw the line *bc* parallel to the axis. Again, from the upper extremity of O draw a line parallel to the axis, cutting the first principal plane at *d*; and from *e*, a point in the second principal plane on the same side of the axis as *d*, and at an equal distance

Fig. 98.—Lens in First Focal Plane of Eye.

from it, draw a straight line *ec* through the second principal focus $F_2$. From *c*, the intersection of the lines *bc* and *ec*, drop a perpendicular on the axis. This will be the image corresponding to the object O.

Let the distance from the first principal focus of the eye to the object be equal to *d*. Then, if $F_1$ is equal to the first focal distance of the eye, it will readily be seen that the magnification *m* is given by the equation—

$$m = \frac{i}{o} = -\frac{F_1}{d}.$$

*Notice that the size of the image is determined by the distance from the axis to the horizontal line bc.*

2. Thin Lens in First Focal Plane of Eye.—The modification produced in the ocular image by placing a lens in the first focal plane of the eye may be readily determined. A straight line can still be drawn from the upper extremity of O through $F_1$, since the lens is thin, and central rays pass through it undeviated. Hence we shall obtain the same line *bc*, to determine the size of the image, as before. Let the lens be convergent, as shown in Fig. 98, and let $f_1$ be its first focal point. Draw a line from $f_1$ through the upper extremity of O, and produce this line to the lens. This ray will be rendered parallel after passing through the lens, and will finally be bent downward, after refraction in the eye, so as to pass through $F_2$. The construction is

similar to that previously employed. The image $I_2$ is thus obtained, which is of the same size as $I_1$, but is brought forward in the eye.

The modification necessary when a divergent lens is used will suggest itself to the student. In that case $f_1$ will be a point on the side of the lens opposite to O.

As a result of this construction, we arrive at the important rule *that a lens placed in the first focal plane of the eye produces no change in the size of the image formed without accommodation, but only shifts the image forward (convergent lens) or backward (divergent lens).*

This rule does not mean that the image *seen by the eye* will be the same with, as without a lens. If the image $I_1$ was formed behind the retina, accommodation would be necessary to bring it forward on to the retina. In accommodation both the posterior and anterior focal distances $F_1$ and $F_2$ are diminished, and therefore the magnification, which is equal to $\frac{F_1}{d}$, will be diminished. If a lens, placed in the first focal plane of the eye, is used to bring the image forward, the dimensions of the latter are unaltered, and it will therefore be larger than that seen by the aid of accommodation. If the eye is normal, the point $F_2$ (Fig. 98), will be on the retina, and thus a retinal image could only be formed without accommodation when the object is situated at the principal focus of the lens.

3. Lens in Front of, or Behind, the First Focal Plane of Eye.—Let the distance from the first focal point $F_1$ to the object O (Fig. 99), be equal to $d$, and let the lens be placed at a distance $x$ from $F_1$ ($x$ will be positive when the lens is farther from the eye than $F_1$). In order to obtain the horizontal line corresponding to $bc$ (Fig. 98), which determines the size of the ocular image, we must first construct the virtual image formed by refraction through the lens. The method of doing this will be seen from Fig. 99, and has already been explained (p. 71). Let $O_1$ be the image formed. From the extremity of $O_1$ remote from the axis, draw a straight line passing through $F_1$, and cutting the first focal plane in $a$. The remaining construction for the ocular image I will be readily understood from Fig. 99.

It is obvious that $P_1a$ will be equal to the size of the ocular image, $\mathbf{i}$ say. Also, since the triangles $AF_1O_1$ and $aF_1P_1$ are similar, we have, if $\mathbf{o}'$ is equal to the size of the image $O_1$,

$$\frac{\mathbf{i}}{\mathbf{o}'} = \frac{-F_1}{v + x}, \quad \ldots \ldots \ldots \quad (1)$$

where $v$ is equal to the distance $LO_1$.

Also, if o is equal to the size of the object O,

$$\frac{o'}{o} = \frac{LO_1}{LO} = \frac{v}{u}, \quad \dots\dots\dots\dots \quad (2)$$

where $u$ is equal to the distance LO.

From (1) and (2) the magnification $m$ of the ocular image is given by

$$m = \frac{i}{o} = -\frac{F_1 v}{u(v+x)}, \quad \dots\dots \quad (3)$$

the minus sign denoting inversion.

FIG. 99.—Lens in front of First Focal Plane of Eye.

Now, $$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad \therefore \frac{u}{v} = 1 + \frac{u}{f} = 1 + \frac{d-x}{f},$$

since $u = LO = FO - F_1L = d - x$.

Also, $$v = \frac{fu}{f+u} = \frac{f(d-x)}{f+d-x},$$

$$\therefore m = \frac{i}{o} = -\frac{F_1}{\left(1 + \frac{d-x}{f}\right)\left(\frac{f(d-x)}{f+d-x} + x\right)}$$

$$= -\frac{fF_1(f+d-x)}{(f+d-x)(fd-fx+fx+dx-x^2)} = -\frac{fF_1}{(fd+dx-x^2)} \dots (4)$$

This gives the magnification in terms of $F_1$, $d$, and $x$.

O

(*a*) Let $x = 0$; then the lens is in the first focal plane of the eye. In this case—

$$m = \frac{i}{o} = -\frac{fF_1}{fd} = -\frac{F_1}{d},$$

so that the size of the ocular image is the same as if the lens were absent, as already proved.

(*b*) Let the lens be *convergent*, and for $f$ write $-f$, since the focal length of a convergent lens is negative. Then—

$$m = \frac{fF_1}{(-fd + dx - x^2)}.$$

Since $d$ will always be greater than $x$, if the lens is placed between $F_1$ and O, we see that $dx - x^2$ will be positive, so that, at first, the numerical value of the denominator decreases as $x$ is increased; thus, *the size of the ocular image at first increases as the lens is moved in advance of the first focal plane of the eye.* The denominator will continue to decrease until $(dx - x^2)$ has reached its greatest value, which will occur when $x = \frac{d}{2}$. Then the denominator will commence to increase. When $x = d$,

$$m = \frac{fF_1}{-fd} = -\frac{F}{d}.$$

That is, when the lens is placed in contact with the object, the ocular image will again have the same size as if the lens were removed.

If the lens is moved from $F_1$ toward the eye, $x$ will be negative, and $(dx - x^2)$ will be negative, so that the denominator will increase numerically, and the size of the ocular image will diminish.

(*c*) Let the lens be divergent, so that $f$ is positive in (4). Then, when $x$ is positive, the denominator of (4) will increase in numerical value, and the size of the ocular image will diminish, as $x$ is increased, *i.e.* as the lens is moved from $F_1$ away from the eye. Similarly, the size of the ocular image will increase as the lens is moved from $F_1$ toward the eye.

**Summary.**—We may now summarise the results obtained.

1. Any lens placed in the first focal plane of the eye makes no difference in the size of the ocular image.

2. If the lens is convergent, the size of the ocular image will increase as the lens is removed from the first focal plane, away from the eye. It will reach its maximum size when the lens is half-way between the first focal plane of the eye and the object.

The ocular image will continually decrease in size as the lens is moved from the first focal plane toward the eye.

3. If the lens is divergent, the size of the ocular image will decrease or increase, according as the lens is moved away from, or toward, the eye, from the first focal plane of the latter. The smallest image will be formed when the lens is midway between the first focal plane and the object.

**Practical Consequences.**—The anterior focus of the eye is at a distance of 13·7 mm. from the cornea. Spectacle-lenses are generally worn in advance of this position, owing to the projecting eyelashes.

In hypermetropia, due merely to deficient length of the eye, a convergent spectacle-lens placed at the anterior focus merely brings the ocular image forward so as to coincide with the retina. If the retinal image thus formed is that of a near object, it will be larger than that formed in an emmetropic eye by the aid of accommodation, and if the spectacles are worn in advance of the anterior focus of the eye, a still greater magnification is produced. On the other hand, the hypermetropic eye, which sees near objects merely by means of accommodation, will have smaller retinal images than the emmetropic eye, since the first focal length of the accommodated eye must be smaller in hypermetropia than in emmetropia, in order to bring the ocular image on to the retina.

In myopia, a divergent lens placed at the first focal plane of the eye serves to throw the ocular image of a near object farther back. If the lens used is adapted for distant vision, the same degree of accommodation will be required for near vision as in the emmetropic eye, and the retinal images will be of the same size in both cases. Since the divergent lens is generally worn in front of the first focal plane of the eye, the retinal image will be diminished. In extreme cases of myopia, this diminution in the size of retinal images is so marked that recourse is sometimes had to extraction of the crystalline lens.

In presbyopia, complete loss of accommodative power in a normal eye can only be remedied by placing the object to be viewed in the anterior focal plane of the convergent glasses worn. The relative positions of eye and object remaining unaltered, moving the glasses forward from the first focal plane of the eye will throw the image behind the retina, and distinct

vision will not be obtained. On the other hand, the image on the retina, though indistinct, will be larger than before. For this reason, presbyopes will often be seen reading with their glasses perched on the extreme tip of the nose.

**Vision through a Magnifying Glass.**—If a convergent lens is placed in front of a normal eye, an object situated at the focus of the lens will be seen without accommodation. If the lens is placed in the first focal plane of the eye, the size of the retinal image will be the same as if the eye had been sufficiently long for an image to be formed without the aid of either lens or accommodation. If the lens is moved away from the eye, or if the object is moved nearer to the lens, accommodation becomes necessary. The largest retinal image will be formed when the lens is in the first focal plane of the eye, and the object is placed at such a distance from it that the virtual image formed by the refraction of the lens shall be at the near point of the eye. This follows from the fact that the magnification, which is equal to $-F_1/d$, will be affected more by the diminution of $d$, as the object is brought nearer to the lens, than by the diminution in $F_1$ produced by accommodation.

Let D be the distance from the first principal focus to the near point of the eye. Then, if a lens of focal length $f$ is placed in the first focal plane of the eye, the distance $u$ from the lens, at which an object must be placed in order that the lens shall form an image at the near point, is given by—

$$\frac{1}{D} - \frac{1}{u} = \frac{1}{f}; \quad \therefore \quad u = \frac{Df}{f - D}.$$

The magnification produced by the combination of the eye and the lens is equal to—

$$-\frac{F_1}{u} = -\frac{F_1(f - D)}{Df}$$

If the object itself were placed at the near point, the lens being removed, the magnification produced by the unaided eye would be equal to—

$$-\frac{F_1}{D}.$$

In both cases the eye must be accommodated to the same extent, and therefore $F_1$ has the same value in both cases. Thus—

*Maximum magnification produced by eye and lens: Maximum magnification produced by unaided eye*—

$$= -\frac{F_1(f-D)}{Df} : -\frac{F_1}{D} = \frac{f-D}{f} : 1 = 1 - \frac{D}{f} : 1.$$

The same result may be arrived at more simply, by noting that the image seen by the aid of the lens, and the object itself when seen by the unaided eye, are both situated at the near point. Thus the eye being similarly accommodated in both cases, the magnification is merely that produced by the lens, which, from equation (10), p. 72, is equal to $-(D-f)/f$. It must be remembered that for a magnifying glass, $f$ is negative.

In the normal eye, D = 25 cm. (about). Thus, a lens of P dioptres, (p. 74), when used as a magnifying glass, produces a magnification equal to—

$$1 + 0{\cdot}25\,P.$$

**Achromatism of Magnifying Glass.**—A small white object, when viewed axially through a magnifying glass, appears to be achromatic. The explanation of this is obvious. When the magnifying glass is placed in the first focal plane of the eye, the ocular images, formed by blue and red rays, are equal in size, but differ slightly in position. Thus the only defect produced is that some of the coloured images are more sharply focussed on the retina than others.

Another explanation can also be given. If the object, of linear dimensions $\mathbf{o}$, is situated at a distance $u$ from the lens, while the blue and red images are formed at distances $v_b$ and $v_r$ from the lens, then the blue image subtends an angle

$$\frac{v_b}{u}\mathbf{o} \div v_b = \frac{\mathbf{o}}{u}$$

at the lens, and practically the same angle at the eye. The red image also subtends an angle equal to $\mathbf{o}/u$, so that the ocular images are equal in size.

When a large white object is viewed through a lens, the marginal parts of the image show traces of colour. This is due to the chromatic effects of spherical aberration (p. 134).

**Pocket Microscopes.**—A powerful lens, when placed near the eye, may be used to magnify a small object placed within the anterior focal distance of the lens. The marginal portions of the image seen will, however, be very indistinct, owing to spherical aberration, unless a stop is used.

WOLLASTON'S LENS consists of two glass hemispheres, their plane sides being cemented together with a stop interposed. Thus, only centric pencils can reach the eye from the object, and aberration is avoided.

THE CODDINGTON LENS consists of a glass sphere round which a deep equatorial groove has been ground, so that it takes the form shown in Fig. 100. In this, as in Wollaston's lens, the light reaching the eye is limited to centric pencils, but the loss of light which occurs in Wollaston's lens, owing to the interposition of a layer of transparent cement of different refractive index from the glass, is avoided. Experience shows that the diameter of the central aperture through which the light passes must be limited to about one-fifth of the focal distance of the sphere. The image seen will be strongly curved, since peripheral portions of the object will be at a greater distance than central portions from the centre of the sphere.

FIG. 100.—Coddington Lens.

THE STANHOPE LENS consists of a glass cylinder, the ends of which are ground convex to unequal radii, so as to diminish spherical aberration. The cylinder is made of such a length that an object placed on the end surface of least curvature will be seen by an eye placed near the opposite, more convex surface.

## QUESTIONS ON CHAPTER IX

1. The maximum distance of distinct vision for a certain person is 20 centimetres. To enable him to see distant objects distinctly, he will require a lens. Calculate either (*a*) the power, in dioptres, of that lens, or (*b*) its focal length, in centimetres. Explain also, with the aid of a diagram, why this lens will enable him to have distant vision.

2. Illustrate by a figure the action of a simple convex lens of 6 inches focal length, placed close in front of an eye whose distance of distinct vision is 14 inches, and find the magnifying power.

3. Find the magnifying power of a simple lens of 1 inch focal length, placed close in front of an eye whose distance of distinct vision is 10 inches. Illustrate with a figure.

## CHAPTER X

### OPTICAL INSTRUMENTS AND APPLIANCES

**Refracting Telescopes.**—The optical system of a refracting telescope, in its simplest form, consists of two coaxial lenses, one of which, termed the **object-glass or objective,** is turned toward a distant object, while the other, termed the **eye-lens,** is placed immediately in front of the eye. A virtual image of the distant object is then seen, and this image subtends at the eye an angle greater than that subtended by the object. It is commonly stated that a telescope magnifies distant objects, and, when understood in the sense explained above, this statement is correct; but it is not at all necessary that the virtual image seen by the eye should be larger than, or indeed as large as, the object.

The objective of a refracting telescope is always a convergent lens; the eye-lens may be either convergent or divergent.

**Astronomical Telescope.**—In this instrument the objective, O (Fig. 101), forms a real, inverted, and diminished image of a

Fig. 101.—Optical System of Astronomical Telescope.

distant object. This image is viewed through a convergent eye-lens, E, which merely acts as a magnifying glass; the image

may therefore be situated anywhere between the first focal point and the surface of the eye-lens, provided that the virtual magnified image is formed at a distance from the eye greater than the shortest distance of distinct vision. The image seen is inverted.

Astronomical telescopes are often provided with fine spider lines perpendicular to the axis of the telescope, and intersecting each other on the axis. These spider lines are termed **cross-wires**; when the plane in which they lie coincides with that in which the real image is formed by the objective, the cross-wires and the image can be clearly seen simultaneously, without parallax. On bringing any part of the image on to the intersection of the cross-wires, we know that the corresponding part of the object lies in the prolongation of the axis of the telescope.

**Galileo's Telescope.**—In this instrument (Fig. 102) the rays which have traversed the objective converge toward points in an inverted diminished image; but before this image is formed,

FIG. 102.—Optical System of Galileo's Telescope.

the rays fall on a divergent eye-lens, E, and are thereby rendered divergent. The image seen on looking through the eye-lens is erect; its formation will be understood from Fig. 102. If the image, toward which the rays from the objective converge, is situated at the first principal focus of the eye-lens, the virtual image finally formed will be at infinity. Galileo's telescope is shorter than an astronomical telescope of equal magnifying power; this circumstance, together with the fact that the image seen is erect, has led to the adoption of Galileo's optical arrangement in the construction of opera glasses.

**Magnification produced by Refracting Telescope.**—Let U be the distance of an object from the objective of a telescope,

and let the rays leaving the objective converge toward an image at a distance from it equal to V. Then, if the length of the object is equal to A, the length of the image will be equal to $\mathrm{AV}/\mathrm{U}$ (p. 72). If the eye-lens is at a distance $u$ from this image, and the final image is at a distance $v$ from the eye-lens, the magnification produced by the eye-lens is equal to $v/u$, and the length of the final image will be equal to $\mathrm{AV}v/\mathrm{U}u$. Since this image is practically at a distance $v$ from the eye, the angle which it subtends at the eye is equal to—

$$\frac{\mathrm{AV}v}{\mathrm{U}u} \div v = \frac{\mathrm{AV}}{\mathrm{U}u}.$$

The object itself will practically be at a distance U from the eye, so that it subtends an angle equal to $\mathrm{A}/\mathrm{U}$ at the eye. Thus, the ratio of the angles which the image and object subtend at the eye is equal to—

$$\frac{\mathrm{AV}}{\mathrm{U}u} \div \frac{\mathrm{A}}{\mathrm{U}} = \frac{\mathrm{V}}{u}.$$

This gives the magnification produced by the telescope.

Since the object is supposed to be at a great distance, the image formed by the objective will practically coincide with the second principal focus of the latter, and $\mathrm{V} = \mathrm{F}$, where F is the focal length of the objective. If it is required that the image shall be seen by the eye without accommodation, it must be formed at the first principal focus of the eye-lens, so that $u = -f$, where $f$ is the focal length of the eye-lens. Thus, **the magnification produced by a telescope, when the latter is adjusted to suit the unaccommodated eye, is equal to** $-\mathrm{F}/f$.

To obtain a high magnification, F must be made as great, and $f$ as small, as possible. Consequently, **a telescope must comprise a long focus objective, and a short focus eye-lens.**

**Terrestrial Telescope.**—A telescope which produces an inverted image is of little use for observing terrestrial objects. This difficulty may, however, be overcome in a very simple manner. Let an objective, O, produce a real inverted image, $I_1$ (Fig. 103). A convergent lens, L, forms an image of $I_1$ at $I_2$, and if the images $I_1$ and $I_2$ are equidistant from, and on opposite sides of, L, they will be equal in size, and will differ only in one being the inversion of the other. It is easily proved that in this

case the distance from $I_1$ or $I_2$ to the lens must be equal to twice the focal length of the latter. The image $I_2$ is observed through an eye-lens, E.

When a single erecting lens, L, is used, it should be equi-convex, to diminish spherical aberration (p. 132). In practice, two similar plano-convex lenses, separated by a distance equal to the focal length of either, are used instead of the single lens L. The image $I_1$ must then be situated at the first principal focus of the lens nearer to it, and the image $I_2$ is formed at the second principal focus of the other lens. The convex surfaces of the lenses face each other, thus ensuring that all

FIG. 103.—Optical System of Terrestrial Telescope.

four surfaces, as far as possible, produce equal increments of deviation. Spherical aberration is minimised by this arrangement, since the deviation is equally divided between four surfaces instead of two.

**The Compound Microscope.**—This instrument, in its simplest form (Fig. 104), consists of an objective, O, which forms a real, magnified, and inverted image of an object placed just beyond its first principal focus; together with an eye-lens, E, by the aid of which this image is viewed. Since the objective produces a magnified image, the object must be nearer to it than the image. For the image seen by the eye to be formed at infinity, the real image due to the objective must be at the first principal focus of the eye-lens; in this case there is no strain on the eye of the observer. The final image may, however, be formed at any distance from the eye-lens exceeding the shortest distance of distinct vision (about 10 inches, or 25 centimetres); in such cases the image due to the objective must be closer to the eye-lens than the first principal focus of the latter.

If the object, of length A, is at a distance U from the microscope objective, and the real image is at a distance V from the objective, the length of the image will be equal to VA/U. The maximum magnification which can be produced by the eye-lens is equal to

FIG. 104.—Optical System of Compound Microscope.

$(1 - D/f)$, where D is the shortest distance of distinct vision, and $f$ is the focal length (a negative quantity) of the eye-lens (p. 196). Thus, when the final image is formed at the near point of the eye, the microscope produces a magnification equal to—

$$\left\{\left(1 - \frac{D}{f}\right) \cdot \frac{VA}{U}\right\} \div A = \left(1 - \frac{D}{f}\right)\frac{V}{U}.$$

The magnification may be increased by separating the objective and eye-lens more widely, thus increasing V; and at the same time bringing the object nearer to the objective, thus decreasing U. When the length of the microscope is considerable, and a high power objective is used, U becomes practically equal to the focal length of the objective, with sign reversed. It thus becomes apparent that **a compound microscope must comprise a short focus objective and a short focus eye-lens.**

**Defects of Telescopes and Microscopes.**—The optical arrangements described above suffer from various defects, partly connected with the objectives, and partly connected with the eye-lenses. These defects arise, for the most part, from two causes: chromatic aberration and spherical aberration.

The method by which a telescope objective is rendered comparatively free from the defects due to chromatic and spherical aberration has already been discussed (p. 95). A low-power microscope

objective consists of a crown glass plano-convex lens—of which the plane face is turned toward the object so as to diminish spherical aberration—backed by a divergent lens of flint glass, designed to correct the chromatic aberration. High-power microscope objectives are exceedingly complicated in structure; one is represented in Fig. 54 (p. 98). The plane face of the lower lens dips into oil, in which the object is immersed; this lens exhibits strong chromatic aberration, which is corrected by the remaining lenses. Prof. Abbe discovered that, by slightly displacing some of these lenses with respect to the rest, a very perfect correction for chromatic aberration can be ensured.

An eye-lens introduces defects, partly due to spherical aberration, and partly due to the chromatic differences in the spherical aberration (p. 134). The manner in which these defects are minimised will be described later.

We must now refer to a defect of an eye-lens arising from an entirely different cause. Rays diverge in all directions from each point of a natural luminous object; but an image, such as that formed by the objective of an optical instrument, is formed by narrow pencils, and each point of it must be seen by means of a narrow pencil. The only points of the image which will be visible, when the eye is in any given position, are those from which pencils can simultaneously penetrate the pupil of the eye. On glancing at Figs. 101, 103, and 104, it will be seen that the pencils from the extremities of the image are refracted through the peripheral portions of the eye-lens, in such positions that they could not simultaneously enter an aperture so small as the pupil, placed near the eye-lens. Thus, those parts of the image which are near the axis will alone be seen, and the field of view will be very limited. This defect is overcome by the use of a field lens.

**The Field-Lens.**—Let OA (Fig. 105) represent the objective of an optical instrument, forming a real image, I. The object of which I is the image is not shown. Let the image I be formed in the principal plane of a lens F. This will make no difference in the dimensions of the image finally seen, since no magnification occurs. On the other hand, the pencils, by means of which I is seen, will be deflected about their points of origin, in such a manner that they cross the axis near the second principal focus of F; thus, pencils from all points of I can simultaneously enter the pupil of an eye placed near the latter

point. If an eye-lens E, of focal length equal to that of F, is placed with its principal plane passing through the second principal focus of F, the whole of the image I will be seen at

FIG. 105.—Illustrates the Function of a Field-Lens.

infinity. The combination of the lenses E and F is termed an **eye-piece**. F is termed the **field-lens**, since it enlarges the field of view; E is termed the **eye-lens**.

The two rays, shown as diverging from one extremity, A, of the objective, will, after refraction through F, converge toward a point *a*, which is the image of A. Similarly, *o* is the image of O, the opposite extremity of the objective, so that *oa* is the image of the objective formed by the field-lens. When the distance between the objective and field-lens is considerable, the image *oa* will be formed practically at the second principal focus of F, *i.e.* in the principal plane of the eye-lens E. All rays from I will pass through the circular space enclosed by the image *oa*.

**Kellner's Eye-piece.**—This eye-piece consists of a combination of lenses identical with that just described. **The field and eye lenses are of equal focal lengths, and are separated by a distance numerically equal to the focal length of either.** It has a very wide field, and is suitable for use with a microscope when wood-sections, &c., are being examined. Since the magnification is entirely produced by the eye-lens, the effects of spherical aberration will be noticeable, the peripheral parts of the field being disproportionately magnified (Fig. 69, p. 130). Kellner's eye-piece is achromatic in the sense that an ordinary magnifying glass is achromatic, *i.e.* the red and blue images are not equal in size,

and are formed at different distances from the eye, but both subtend the same angle at the eye (p. 197). Perfect achromatism may be secured by using a crown and flint combination, similar to an achromatic objective (p. 95) for the eye-lens. An important disadvantage of Kellner's eye-piece arises from the circumstance that the image I and the surface of the lens F are simultaneously in focus, so that smears or dust on the surface of the lens F are obtrusively visible.

**Ramsden's Eye-piece.**—In many instruments it is necessary to observe the coincidence of a point of the image with the intersection of cross-wires, or to measure the dimensions of the image by the aid of a scale in the eye-piece. Ramsden's eye-piece was designed to meet these requirements. **It consists of two lenses, of equal focal lengths, separated by a distance equal to two-thirds of the numerical value of the focal length of either.**

Let F be the focal length of the equivalent lens (p. 74); this may be termed the focal length of the eye-piece (Question 10, p. 158). Then, if $(-f)$ is the common focal length of the eye and field lenses—

$$\frac{1}{F} = -\frac{1}{f} - \frac{1}{f} + \frac{2}{3}\frac{f}{f^2} = -\frac{2}{f} + \frac{2}{3f} = -\frac{4}{3f}.$$

$$\therefore\ F = -\frac{3}{4}f.$$

The equivalent lens must be placed behind the field-lens, at a distance from the latter equal (p. 76) to—

$$\frac{F}{f} \times \frac{2}{3}f = \frac{3f}{4f} \times \frac{2}{3}f = \frac{1}{2}f.$$

Thus, the equivalent lens must be placed between the field and eye lenses. For an object to be seen at infinity, it must be situated at the first principal focus of the equivalent lens, or at a distance in advance of the field-lens equal to—

$$\frac{3f}{4} - \frac{f}{2} = \frac{1}{4}f.$$

This is the position at which the cross-wires must be placed, and at which the image due to the objective must be formed (Fig. 106).

In Ramsden's eye-piece the field-lens is less efficient than in Kellner's arrangement, but the field is fairly wide. Chromatic effects, though not entirely absent, are not very obtrusive. In some cases

each of the lenses consists of a crown and flint combination, when chromatic effects are eliminated. Since the image is seen through two lenses, the deviation of a ray is produced in four increments, each of which is small. Thus, the spherical aberration produced is small, and is further diminished by making both lenses plano-convex, their convex surfaces facing each other (Fig. 106). Consequently, the image seen is fairly free from distortion.

FIG. 106.—Optical System of Ramsden's Eye-piece.

**Huyghens's Eye-piece.**—In designing this eye-piece, Huyghens sought to diminish the effects of spherical aberration as much as possible. To attain this end, he chose the focal lengths and positions of the field and eye lenses, so that each lens produces an equal increment of deviation in a ray initially parallel to the axis (compare p. 132).

Let $P_1Q$ and $P_2R$ (Fig. 107) be the respective principal planes of the field and eye lenses. Let AB be a ray incident on the field-lens, in a direction parallel to the axis LM. The corresponding refracted ray BC is directed toward F, the second principal focus of the field-lens. Produce AB to E. Then it is obvious that the deviation produced by the field-lens is equal to $\angle EBC = \angle CFP_2$.

FIG. 107.—Illustrates the conditions to be complied with by Two Lenses, in order that Spherical Aberration shall be minimised.

Let the ray BC, incident at C on the eye-lens, give rise to a refracted ray CG, cutting the axis at G. Then the deviation produced by the eye-lens is equal to $\angle FCG$, and if the lenses produce equal deviations, $\angle CFG = \angle FCG$, and therefore FG = CG. When the total deviation is small, $P_2G$ will be very nearly equal to CG, and therefore $P_2G = GF = P_2F/2$.

Now the ray BC is directed toward the point F on the axis, and

the corresponding refracted ray cuts the axis at G; hence, it follows that the points F and G are conjugate foci with respect to the eye-lens, and if $P_2F = u$, then $P_2G = v = u/2$. If $f_2$ is the focal length of the eye-lens—

$$\frac{1}{u/2} - \frac{1}{u} = \frac{1}{f_2}; \therefore \frac{1}{u} = \frac{1}{f_2}.$$

$$u = f_2.$$

Thus, F must be the second principal focus of the eye lens $P_2R$. But F is also the second principal focus of the field-lens $P_1Q$. Thus, if $P_2P_1$, the distance between the lenses, is equal to $d$ (a positive quantity)—

$$d = FP_1 - FP_2 = -f_1 - (-f_2) = f_2 - f_1 \cdot \quad . \quad . \quad . \quad (1)$$

This gives the condition that the field and eye lenses shall produce equal increments of deviation in a ray initially parallel to the axis.

Huyghens arranged that **the focal lengths of the field and eye lenses are in the ratio 3 : 1, while the distance between them is numerically equal to twice the focal length of the eye-lens.** Thus, if $f_2 = -f$, $f_1 = -3f$, and $d = -f + 3f = 2f$.

The focal length F of the equivalent lens is given (p. 74) by—

$$\frac{1}{F} = -\frac{1}{3f} - \frac{1}{f} + \frac{2f}{3f^2} = -\frac{2}{3f}.$$

$$\therefore F = -\frac{3}{2}f.$$

The equivalent lens must be placed at a distance behind the field-lens (p. 76) equal to—

$$\frac{F}{f} \times 2f = \frac{\frac{3}{2}f}{f} \times 2f = 3f.$$

Hence, the equivalent lens must be placed behind the eye-lens, at a distance $3f - 2f = f$ from the latter.

The first principal focus of the equivalent lens (or the first principal focus of the eye-piece) is at a distance $3f/2 - f = f/2$ in front of the eye-lens. This point, which lies between the eye-piece lenses, gives the position at which the image due to the objective must be formed, in order that the final image may be seen at infinity. Thus, the image due to the objective must be formed on the negative side of the field-lens; for this reason **Huyghens's eye-piece is termed a negative eye-piece.** On the other hand, **Ramsden's eye-piece is termed a positive eye-piece.**

The image seen by the eye will be practically free from distortion, since the spherical aberration is exceedingly small. To still further decrease the effects of spherical aberration, both lenses are generally made convexo-plane, the convex surfaces facing the incident rays (Fig. 108). To obtain absolutely the best possible results, Airy recommends that the field-lens should be convexo-concave, its radii of curvature being in the ratio 4 : 11, the convex surface (which is the one most strongly curved) facing the incident rays; the eye-lens should be bi-convex, its radii being in the ratio 1 : 6 (crossed lens, p. 133), the more strongly curved surface facing the incident rays.

It may be noticed that the image I (Fig. 108), due to the objective, will be curved (p. 129), its radius of curvature being positive.

FIG. 108.—Optical System of Huyghens's Eye-piece.

The field-lens forms a real image, midway between E and F. This image will be distinct, since each point of it is formed by a narrow excentric pencil refracted through a limited area of the field-lens. It will, however, be curved and distorted, its peripheral parts being disproportionately compressed (p. 131).

The image seen by the eye (not shown in Fig. 108, since it is formed at infinity), will be distinct but curved, its radius of curvature being positive. Distortion is eliminated, since the eye-lens forms a magnified virtual image, its peripheral parts being disproportionately magnified, thus correcting the disproportionate compression of the peripheral parts of the image formed by the field-lens.

If cross-wires are required to be used with Huyghens's eye-piece, they must be placed midway between the eye and field lenses, at the position occupied by the real image formed by the field-lens. Trustworthy measurements of the dimensions of

the object cannot be obtained by placing a scale, or movable cross-wires, in the position mentioned; the scale would be seen only through the eye-lens, and the divisions remote from the axis would be disproportionately magnified. Ramsden's eye-piece was designed to avoid this defect.

**Achromatism of Huyghens's Eye-piece.**—For reasons explained on p. 101, an eye-piece composed of two simple lenses can never be *truly* achromatic, since the red and blue images occupy different positions, and are unequally magnified. On the other hand, the effects of chromatic aberration will not be seriously prejudicial to an eye-piece if the red and blue images subtend equal angles at the eye. In this case the eye-piece is achromatic, in the sense that an ordinary magnifying glass is achromatic. It was proved by Boscovich that Huyghens's eye-piece is achromatic in this sense.

The condition that the red and blue images formed by an eye-piece shall subtend equal angles at the eye may be investigated as follows. It will be evident, from an inspection of the various figures in the preceding part of this chapter, that the rays falling on the field-lens are all nearly parallel to the axis. This is due to the circumstance that the objective subtends only a small angle at the image which it forms, and this image is small; thus each pencil is narrow, and only slightly inclined to the axis. Let AB (Fig. 109) be a ray of white light parallel to the axis, incident at a point B on the field-lens $P_1Q$. Let the red refracted ray BC fall on the eye-lens $P_2R$ at a point C; then the blue refracted ray BG will be more deviated, and will fall on the eye-lens at a point G, nearer to the axis than C. Let CD be the red ray emerging from the eye-piece, while GH is the corresponding blue ray. Produce DC to E, and HG to K. Let AB be a ray

FIG. 109.—Illustrates the condition to be complied with by an Eye-piece, in order that the Red and Blue Images seen shall subtend Equal Angles at the Eye.

proceeding toward the upper extremity of the image formed by the objective; then the upper extremity of the red image formed by the eye-piece will be situated at a distant point on the line CE, and the upper extremity of the blue image will be situated at some other distant point on the line GK. Thus, if the eye is placed near D, the red image will subtend at it an angle, EDM, while the blue image will subtend an angle KHM. For the red and blue images to subtend equal angles at the eye, we must have $\angle$ EDM = $\angle$ KHM, or the lines CD and GH must be parallel. Since the ray AB is parallel to the axis, the angles EDM and KHM measure the resultant deviations produced in the corresponding red and blue rays, and thus we see that **for an eye-piece to be achromatic,** in the sense defined above, **red and blue rays, initially parallel to the axis, must be equally deviated.**

The blue ray is more deviated than the red ray at B; the blue ray must be less deviated at G than the red ray is at C, a condition which is possible, since the point G is nearer than C to the axis of the eye-lens.

In Fig. 107, $\angle CGP_2$ measures the deviation of the ray AB; let us suppose that this ray is of a colour intermediate between red and blue. Let $f_1$ and $f_2$ be the focal lengths of the field and eye lenses for rays of this intermediate colour; and let $P_2P_1 = d$. Since the angle $CGP_2$ is supposed to be small, it may be measured by its tangent, *i.e.* $\angle CGP_2 = P_2C/GP_2$. Let $P_1B = y$. Then—

$$P_2C/P_1B = FP_2/FP_1 = (FP_1 - P_2P_1)/FP_1.$$

$$\therefore\ P_2C = y\,.\,\frac{(-f_1 - d)}{-f_1} = y\frac{(f_1 + d)}{f_1}.$$

Also—

$$\frac{1}{P_2G} - \frac{1}{P_2F} = \frac{1}{f_2}.$$

$$\therefore\ \frac{1}{P_2G} = \frac{1}{f_2} + \frac{1}{f_1 + d} = \frac{f_1 + f_2 + d}{f_2(f_1 + d)}.$$

And—

$$\angle CGP_2 = \frac{P_2C}{GP_2} = P_2C \times \left(\frac{1}{-P_2G}\right) = -y\,.\,\frac{f_1 + f_2 + d}{f_1 f_2}.$$

But. from p. 76, if F is the focal length of the lens "equivalent" to the combination considered—

$$F = \frac{f_1 f_2}{f_1 + f_2 + d}.$$

Thus, the deviation produced in the ray AB (Fig. 107) is equal to $(-y/F)$. Returning to Fig. 109, we see that $y$ ($= P_1B$) has the same value for all of the coloured rays which jointly constitute the white ray AB. Hence, in order that the coloured constituents of a white ray should suffer equal deviations in passing through the eye-piece, F must have the same value for rays of all colours. As proved on p. 100, this condition is satisfied if—

$$d = -(f_1 + f_2)/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In Huyghens's eye-piece $f_1 = 3f_2 = -3f$, while $f_1 = -f$, and $d = 2f$. Therefore (2) is satisfied.

It must be repeated that an eye-piece satisfying (2) has no advantage, *so far as ordinary chromatic aberration is concerned*, over a single lens used as a magnifying glass. But we have seen (p. 204) that a single lens cannot satisfy the conditions requisite for an eye-piece Further, the image seen through Huyghens's eye-piece, unlike that seen through a single lens, exhibits scarcely any colour near the edges of the field, since equation (1), p. 208, is satisfied, spherical aberration is minimised, and therefore the chromatic effects of spherical aberration (p. 134) are practically absent.

When the objective of an optical instrument is over-corrected for chromatic aberration, the blue image is formed at a greater distance from it than the red image. If this over-correction is properly adjusted, and Huyghens's eye-piece is used, it can be arranged that the field-lens forms a blue image at the principal focus of the eye-lens for blue rays, and a red image at the principal focus of the eye-lens for red rays. In this case the instrument *as a whole* is achromatic in the wider sense, that all the coloured images are formed at infinity, and are approximately equal in size.

**Hadley's Sextant.**—This is an instrument used in measuring the angle subtended, at the eye of the observer, by the line joining two distant objects. By its aid sailors are enabled to measure the altitude of the sun, and thus to determine the latitude.

The sextant consists of a rigid frame-work carrying two glass mirrors, M and N (Fig. 110), and a telescope, T. The telescope is fixed, its axis being parallel to the plane of the paper; it is provided with cross-wires and a Ramsden's eye-piece. The mirror N, which is perpendicular to the plane of the paper, is fixed

with its surface inclined to the axis of the telescope at an angle of 60°. One half of the mirror N is silvered, the remaining half being left clear. Through the clear part of N a distant object can be directly observed by the aid of the telescope. The mirror M, perpendicular to the plane of the paper, is completely silvered, and is capable of rotation about an axis perpendicular to the plane of the paper. It is rigidly connected to an arm MV, which ends in a vernier, V, working over a scale; by the aid of this vernier, rotations of M can be accurately measured. When the arm MV lies along the line MQ, the mirrors M and N are parallel. In that case a ray, AM, parallel to the axis, BT, of the telescope, and incident, at an angle of 30°, on M, is reflected along MN, so as to be incident on the silvered surface of N at an angle of 30°, whence it is reflected along NT, the axis of the telescope. A ray, BT, can also pass through the unsilvered surface of N, along the axis of the telescope. Thus, if the rays AM and BN proceed from a distant object, such as a star, two coincident images of this object are seen on looking through the telescope.

FIG. 110.—Diagrammatic Representation of Hadley's Sextant.

If the arm MV is now turned through an angle OMV, the mirror M is rotated through the same angle. An incident ray, CM, is now reflected along MN, so as ultimately to travel along the axis of the telescope. If there is a distant object on the line MC produced, an image of this will be seen in the telescope, coinciding with the object seen directly along TB. Produce CM to cut the axis of the telescope in D. Then the two objects which give rise to coincident images in the telescope, subtend an angle CDB = CMA at the eye of the observer.

It can easily be seen that $\angle CMA = 2 \times \angle OMV$. For, if we reverse the ray NT, it will follow the path NM, to be reflected along MA when the arm lies along MO, and to be reflected along MC when the arm has the position MV. If $i$ is the angle of incidence when the arm lies along MO, $\angle NMA = 2i$. Let $\angle OMV = \theta$. Then, when the arm has the position MV, the angle of incidence of the ray NM must be equal to $(i + \theta)$, and $\angle NMC = 2(i + \theta)$. Therefore $\angle CMA = 2\theta = 2 \times OMV$. Accordingly, the scale of the instrument is divided into degrees, and each degree is numbered as two degrees.

In observing the altitude of the sun at sea, the instrument is held in a vertical plane, the horizon line is observed directly along the line TB, and the arm MV is rotated until one edge of the sun's disc, as seen in the telescope, coincides with the horizon line. One or more black glasses are introduced between M and N, to diminish the brightness of the sun's image. On shore, where there is no definite horizon line, an **artificial horizon** is used. This is a vessel containing a liquid, generally mercury; the angle subtended at the eye of the observer by the line joining the sun and its reflected image in the mercury, is then measured, and this is equal to twice the angular altitude of the sun.

**The Photographic Objective.**—This is a system of lenses used to throw a real image on to the sensitised film of a photographic plate. The rays most active in producing the photograph correspond to the violet end of the spectrum, while those which affect the eye most strongly correspond to the yellow and yellowish-green parts of the spectrum. Hence, in order that the image may be properly focussed by eye, it is necessary that the lens shall be corrected for chromatic aberration, the violet and yellow rays being brought to a focus at the same point. To attain this end, each component of the lens system consists of a convergent crown glass lens combined with a divergent flint glass lens.

Spherical aberration must also be eliminated as far as possible. For landscape photography, a single compound lens, similar to Fig. 111, is used, a diaphragm with a central circular aperture being placed in front of it. In this case very sharp focussing is not necessary, and indeed is impossible, since the objects photographed are at very different distances. Further, a small amount of distortion will produce no very harmful effects.

For the photography of buildings, it is necessary to use a lens much more carefully corrected for spherical aberration, so as to avoid the distortion described on p. 131. Such a lens is termed **orthoscopic** or **rectilinear**; it generally comprises two compound lenses separated by an appreciable distance, with a perforated diaphragm, or stop, interposed between them (Fig. 112).

FIG. 111.—Objective for Landscape Photography.

**The Magic Lantern.**—From an optical point of view, the magic lantern consists of a series of lenses, arranged to throw on a screen an enlarged image of a photograph or drawing on glass. The lens system which forms the image on the screen (*f*, Fig. 113) is termed the **focussing lens**; it generally consists of two compound lenses separated by an appreciable distance, and is, in its more salient features, similar to a rectilinear photographic objective. The lantern slide *s* is placed between the focussing lens and the source of light *l*. The

FIG. 112.—Rectilinear Photographic Objective.

FIG. 113.—The Magic Lantern.

source of light is generally of small dimensions, and of the rays which radiate from it, only those which pass through the central part of the slide would reach the focussing lens, were it not for the **condenser** *cc*. The function of the condenser is merely to deflect the more divergent rays from the source, so that, after passing through the peripheral parts of the lantern slide, they fall on the focussing lens, and ultimately on the screen. Thus, the condenser enlarges the field, and plays a part similar to that of the field-lens in an eye-piece. Since the condenser has nothing to do with focussing the image on the screen, it need not be corrected for spherical or chromatic aberration. It usually consists of two plano-convex lenses with their convex surfaces in contact.

FIG. 114.—Optical System of Herschel's Reflecting Telescope.

FIG. 115.—Optical System of Newton's Reflecting Telescope.

**Reflecting Telescopes.**—All reflecting telescopes agree in forming an image by means of a concave mirror of large radius of curvature; they differ only in the method of observing this image. In **Herschel's telescope** (Fig. 114), the axis of the mirror is slightly inclined to the incident rays, and the image, I, is

FIG. 116.—Optical System of Gregory's Reflecting Telescope.

thus thrown to one side of the axis, and observed directly by means of an eye-piece. In **Newton's telescope** (Fig. 115), the image is thrown to one side, at right angles to the axis, by means of a small plane mirror, N, or a totally reflecting prism. In **Gregory's telescope** (Fig. 116), the concave mirror is pierced with a small central aperture; it forms an image, I, at a point on the axis between the centre of curvature and the principal focus of a small concave mirror, N, and a secondary magnified image is formed at I′, which is viewed through the aperture in the large mirror by the aid of an eye-piece. **Cassegrain's telescope** (Fig. 117) agrees with Gregory's telescope in having a central aperture through which the final image is viewed, but the small reflector N is convex.

Fig. 117.—Optical System of Cassegrain's Reflecting Telescope.

Reflecting telescopes can be made of much wider aperture than refracting telescopes. When used to view stars, spherical aberration is entirely eliminated by making the concave reflector in the form of a paraboloid of revolution (p. 43); chromatic aberration is, of course, once for all avoided. The image seen is not generally so bright as that obtained by a refracting telescope. The concave reflector is sometimes made of speculum metal, an alloy of copper and tin; a better plan is to grind the concave surface in glass, and then silver this surface and polish it. The surface can then be repolished without so much fear of altering its form; when necessary, the silver can be dissolved off by acid, and a fresh silvering and polishing can then be performed.

## Questions on Chapter X

1. A microscope is made up of an objective of ½ inch focal length and an eye-piece of 1 inch focal length placed 6 inches apart. A person uses it to look at a small arrow-shaped object, his distance of distinct vision being 8 inches. Where must the object be placed?

Draw a careful figure showing the successive images and the

complete course of the pencil by which a point in the object is seen.

2. Describe the construction of a refracting astronomical telescope, and show, by means of a diagram, the course of a pencil of parallel rays during its passage through the instrument.

3. You are given two convex lenses of 23 inches and 1 inch focal length respectively. Explain how to arrange them to form a telescope. Draw a diagram showing the passage of a pencil of rays from a distinct point through the instrument, and calculate its magnifying power when viewing a distant object.

4. What is the usual arrangement of lenses in an achromatic object-glass for a telescope?

State the conditions, and discuss the question, of the achromatism of such lenses.

5. Explain the principle of the opera-glass, drawing carefully the course of rays from a star through it, and into an eye.

6. In a telescope 30 feet long, what would be the focal length of an eye-piece that would give a magnification of 500 diameters?

Explain the advantage of a telescope over a pair of simple sights for purposes of angular measurement.

7. The object-glass and the eye-lens of a compound microscope are each 1 inch in focal length, and the distance between them is 9 inches. Draw a careful diagram showing the passage of a pencil of rays through the instrument, and calculate where the object must be to give distinct vision to a person with normal eyesight.

8. Explain the condition which must be fulfilled for (1) an object-glass, (2) an eye-piece, to be achromatic. Give a carefully drawn diagram showing the passage of a pencil of rays through an achromatic eye-piece.

9. Describe, and point out the respective merits of, Ramsden's and Huyghens's eye-pieces.

10. Describe a sextant, and explain how it enables one to measure the angle subtended, at the observer's eye, by two objects.

### PRACTICAL

1. Arrange the two given lenses to form a microscope, and calculate the position and size of the image of a small object looked at through the arrangement adopted. Draw a careful figure to scale.

2. Select two of the given lenses and arrange them to form a microscope to magnify the given small object. Sketch the arrangement, showing the path of the rays from the object to the eye, and the position of the image.

## CHAPTER XI

### VELOCITY OF LIGHT

**Introductory.**—Between the emission of light from a luminous source, and its arrival at a point at a moderate distance from the same, the interval of time is so short that the propagation of light appears to be instantaneous. Galileo attempted to determine the velocity of light in the following manner. Two observers were stationed at a considerable distance apart, with lamps which could be covered up. One observer uncovered his lamp, and the second observer uncovered his as soon as possible after seeing the light from the first observer's lamp. If this latter operation could be performed instantaneously, the interval of time, noted by the first observer, between the uncovering of his lamp and the observation of the light from the lamp of the second observer, would give the time required for light to travel over twice the distance between the observers. The velocity of light is, however, very great, and the time it requires to travel over any terrestrial distance is consequently so small, that observations of this kind lead to no trustworthy results.

**Römer's Method.**—Astronomical observations of Jupiter's satellites show that while the earth, in its orbital motion, is receding from Jupiter, the mean period between two successive eclipses of a particular satellite is longer than that which elapses when the earth, in its orbital motion, is approaching Jupiter. Römer explained this anomaly on the principle, that when the earth is receding from Jupiter, the light from a disappearing satellite has to travel a greater distance at each

successive disappearance. He also obtained the first trustworthy value for the velocity of light.

Jupiter has five satellites, which revolve in periods lying between 11 h. 58 m. for the satellite nearest to the planet, and 16 d. 16 h. 32 m. 11 s. for the most remote satellite. The time of revolution is measured between successive passages of a satellite through the straight line joining the centres of the Sun and Jupiter. Since the satellites revolve in orbits nearly parallel to the plane of Jupiter's orbit, each satellite, once in every revolution, enters the shadow cone thrown by Jupiter, and so becomes eclipsed. Jupiter itself completes one revolution round the Sun in 11·86 years.

FIG. 118.—Römer's Method of determining the Velocity of Light.

Römer's method of determining the velocity of light from observations on the eclipses of Jupiter's satellites will be understood on referring to Fig. 118. At a certain period in the earth's annual revolution, the earth $E_1$ and Jupiter $J_1$ will be in conjunction. Let us suppose that one of Jupiter's satellites disappears in the shadow of the planet when the latter is at $J_1$. If light were transmitted instantaneously, the actual eclipse, and the observation of the same on the earth at $E_1$, would occur simultaneously. But the light leaving the satellite at the instant of its eclipse has to travel over a distance $J_1 E_1$ before reaching the earth. Let R and $r$ be the respective radii of the orbits of Jupiter and the earth. Then $J_1 E_1 = (R - r)$, and the time required by light, travelling at a velocity V, to cover this distance, will be equal to $(R - r)/V$. Thus, the eclipse of the satellite will be observed on the earth $(R - r)/V$ seconds after its actual occurrence.

After the lapse of 0·545 of a year, the earth $E_2$ and Jupiter $J_2$ will be in opposition. Let the $n$th eclipse of the same satellite occur at this

time. This will be observed on the earth, $(J_2 E_2)/V = (R + r)/V$ seconds after its actual occurrence. If $t$ is the period of revolution of the satellite, the time which has actually elapsed between the first and the $n$th eclipse will be equal to $(n - 1)t$, and the time $T_1$ which has elapsed between the observations of these eclipses on the earth will be equal to $\{(n - 1)t + (R + r)/V - (R - r)/V\} = \{(n - 1)t + 2r/V\}$.

After another period equal to 0·545 year the earth and Jupiter will once more be in conjunction at $E_3$ and $J_3$. In this period $(n - 1)$ revolutions of the satellite will have been completed, and $n$ eclipses will have occurred, the first when the earth and Jupiter were at $E_2$ and $J_2$, and the last when the earth and Jupiter were at $E_3$ and $J_3$. The first eclipse was observed $(R + r)/V$ seconds after its actual occurrence, and the last $(R - r)/V$ seconds after its actual occurrence. Hence the period, $T_2$, between the observations of the first and last eclipses will be $\{(n - 1)t - (R + r)/V + (R - r)/V\} = ((n - 1)t - 2r/V)$ secs.

Römer measured $T_1$ and $T_2$, and found that $T_1 - T_2 = 33$ mins. = 1980 secs. But $T_1 - T_2 = \{(n - 1)t + 2r/V\} - \{(n - 1)t - 2r/V\} = 4r/V$.

$$\therefore \frac{4r}{V} = 1980 \, ; \therefore V = \frac{4r}{1980}$$

Assuming $r$, the mean distance between sun and earth, to be $92 \cdot 8 \times 10^6$ miles, we find that V = 187,000 miles per second, or about 301,000,000 metres per sec.

It will be noticed that this method involves the determination of the time required for light to travel across the earth's orbit, a distance of $195 \cdot 6 \times 10^6$ miles. The time required for this journey is 16·5 mins.

**Aberration of Light.**—Let us suppose that a shot is fired at right angles into a ship (Fig. 119). If the ship is stationary, the shot, if it passes through the ship, will leave it at a point as far behind the bow as that at which it entered. But suppose that, in the period required for the shot to travel from one side of the ship to the other, the ship has moved on through a definite distance. Let AB (Fig. 119) be the position of the ship when the shot strikes it, and let A′B′ be its position

Fig. 119.—To explain the Aberration of Light.

when the shot leaves its opposite side. Then the shot enters the ship at a point distant $a$ from the bow, and leaves it at a point distant $b$ from the bow, where $b > a$. Thus, to an observer on the ship, the shot will appear to have travelled in the direction CD, which is the line joining the two holes made by the shot.

Let V be the velocity of the shot, while $v$ is the velocity of the ship, at right angles to the true path of the shot. Let a time $t$ be required for the shot to pass through the ship, (supposing that V meanwhile remains constant). Then $b - a = vt$, and the distance traversed by the shot, parallel to the direction of its absolute motion, will be $Vt$. Thus, if $\theta$ is the angle of inclination between the apparent and true paths of the shot, $\tan \theta = vt / Vt = v / V$.

When light reaches us from a star, the apparent direction of its course will be affected by the motion of the earth. If $v$ is the velocity of the earth at right angles to the direction of the incident light, which travels with a velocity V, the apparent direction of the light rays will make an angle $\theta$ with their true direction, where $\tan \theta = v/V$.

The orbital velocity of the earth is about 0·0001 times the velocity of light, and the *direction* of the earth's motion is continually changing. Thus, the earth is moving in diametrically opposite directions at the spring and autumn equinoxes. On observing the position of a star near the zenith at these times, it will be seen to occupy different positions with respect to the horizon, since the apparent paths of the light will be inclined equally on opposite sides of the true path; this follows from the circumstance, that if at the spring equinox, $\tan \theta = v / V$, at the autumn equinox $\tan \theta = - v / V$.

This variation in the apparent position of a star, due to the motion of the observer with the earth, is an example of an extensive series of phenomena classified under the head of the **aberration of light**. From observations of the apparent positions of a star at different times of the year, Bradley determined the velocity of light in terms of the known orbital velocity of the earth. The value he found was 308,300,000 metres per second.

**Fizeau's Experiment.**—The first determination of the velocity of light, in terms of measurements confined to the surface of the earth, was effected by Fizeau in 1849. The general principle of his method was to emit light from a small aperture which was

opened and closed many times a second, and to reflect this light back, after it had travelled a considerable distance, to the point from which it started. If light were transmitted instantaneously, it would always, no matter how far it had travelled, be able to enter the aperture from which it had emerged. But if, on the other hand, light is transmitted with a finite velocity, and the closing of the aperture is properly timed, then after travelling to and from a point at a sufficient distance, the returning light will arrive only to find the aperture closed. The resemblance to Galileo's method is sufficiently evident, but in view of the great velocity of light, it is obvious that purely mechanical means must be used to open and close the aperture sufficiently quickly.

Fizeau's apparatus is represented diagrammatically in Fig. 120. A pencil of light from a source S traverses an achromatic system of lenses, and, after reflection at an angle of 45° from a glass

FIG. 120.—Fizeau's Experiment.

plate with plane and parallel faces, converges to a focus at F. The point F is the principal focus of the achromatic object-glass of a telescope, so that the light from the image at F, after traversing this object-glass, is rendered parallel. It then travels a considerable distance (3 or 4 miles) and falls on an achromatic lens L, and is once more brought to a focus, this time on the surface of a concave mirror R. The centre of curvature of this mirror is at the centre of the lens L; thus, the central ray of the convergent cone, formed by refraction through L, will always fall normally on the surface of the mirror, even though its direction should be inclined to the axis of the mirror. The light, after reflection at R, will again be rendered parallel by refraction through L, and will finally form a real image at F. This

image is viewed through the glass plate, by the aid of an eye-piece E.

A toothed wheel is arranged so that, as it rotates, its teeth pass, one after another, through the point F. Thus, the light passing through F is alternately intercepted by a tooth, and allowed to pass between two teeth. When the wheel is rotating slowly, the light from S which has escaped between two teeth, will have sufficient time to travel to R, to be reflected back, and to form a real image at F, before the wheel has sensibly moved. Looking through the eye-piece E, a flickering luminous image will be seen; if the images at F succeed each other more than 8 or 10 times a second, the flickering ceases, owing to the persistence of visual impressions. Now let the speed of the wheel be increased. At a certain speed the light from S which has escaped between two teeth, and after reflection at R, has returned to F, will be intercepted by a tooth which has in the meantime moved through a distance equal to half of that between two teeth. In these circumstances the image previously seen through the eye-piece will disappear. If the speed of the wheel is known, the time required for a tooth to move through this distance will also be known, and since the light has, in this time, travelled from F to R and back again, the velocity of light becomes known.

If the speed of the wheel is still further increased, the light from S which has escaped through a space between two teeth, will form an image in the next space, so that the luminous image reappears. At a still greater speed, the returning light will once more be intercepted by a tooth, the latter having moved through one and a half times the distance between two teeth, while the light travelled from F to R and back again. In a word, the luminous image appears and disappears periodically as the speed of the wheel increases.

Experimenting in the manner described, Fizeau obtained 315,000,000 metres per second as the velocity of light.

**Advantages and Disadvantages of Fizeau's Method.**—

ADVANTAGES.—(1) The principle of the method is quite straightforward, and involves no assumptions which cannot be readily justified.

(2) The image seen through the eye-piece corresponds exactly to the image of S originally formed at F.

DISADVANTAGES.—(1) The light from S is greatly weakened by reflection at the glass plate, so that the image seen through the eye-piece is very faint.

(2) The light from S, when intercepted by a tooth, is reflected back toward the eye, so that there will be a certain amount of general illumination in the field of the eye-piece. This can be avoided by bevelling the teeth so that the light is reflected toward the side of the telescope.

(3) As the speed of the wheel is increased, some time will elapse between the disappearance and the reappearance of the image. Thus, the particular speed at which the returning light is intercepted by the *middle* of a tooth is uncertain.

**Improvements on Fizeau's Method.**—Between 1874 and 1878 **M. Cornu** repeated Fizeau's experiment with greatly improved apparatus. The rotations of the wheel were automatically recorded, so that its speed *at any instant* could be determined with accuracy. Observations were made of the speed of the wheel when the brightness of the image fell to a certain value, and again when, after disappearing, it reached the same value. Thus, disadvantage (3) above was overcome. The distance FR (Fig. 120) was increased to about 15 miles. Cornu's final results indicated that the velocity of light lies between 300,100,000 and 300,700,000 metres per second.

In 1880–1, **Young and Forbes** introduced some novel features into a repetition of Fizeau's experiment. To overcome disadvantage (1), the surface of the glass plate on which the light from S (Fig. 120) is incident, was silvered and polished, a small aperture being left through which the image at F could be viewed. The teeth were bevelled, so as to overcome disadvantage (2). The parallel pencil leaving the telescope fell partly on the collimator LR (Fig. 120), and partly on a similar collimator L′R′ placed behind, and slightly to one side of, LR. The distances FR and FR′ were in the ratio of 12 to 13. Two small luminous images were seen near to each other at F, and as the rays forming these had travelled different distances, they did not vanish simultaneously. Observations of the speed of the wheel were made when the two images were equal in brilliancy. The value found for the velocity of light was 301,382,000 metres per sec.

It was noticed that as the speed of the wheel increased or diminished, the image which was increasing in brilliancy was blue, while that which was diminishing in brilliancy was red. Young and Forbes concluded that the blue rays of the spectrum are propagated at a velocity about 1·8 per cent. greater than the red rays. This conclusion is at variance with the results obtained by all other experimenters. If it were correct, one of Jupiter's satellites would appear to be red at the

moment of its disappearance during an eclipse, and blue on its first reappearance. The stars would also appear as small spectra due to the aberration of light. These discrepancies induce grave doubts as to the advantage of the distinctive features introduced by Young and Forbes.

**Fizeau and Foucault's Method.**—In 1838 Arago proposed the use of a rotating mirror, such as had previously been employed by Wheatstone to determine the duration of the electric spark and the so-called "velocity of electricity," for the purpose of determining the velocity of light. He also pointed out that by its aid the velocity of light in air and in water could be determined, and the rival theories of the nature of light (p. 235) could thus be brought to a crucial test. Under the counsel of Arago, Fizeau and Foucault together designed an arrangement to carry out this investigation. Their partnership, however, came to an end before the minor details of the method were finally settled. On the 6th of May, 1850, both physicists presented independent reports to the Academy of Sciences. The features of their optical arrangements were practically identical; but while Foucault had obtained decisive evidence that light is transmitted more slowly in water than in air, Fizeau had been prevented, by an accident, from reaching this stage of the investigation.[1] It is plain, however, that the name of Fizeau should be associated with that of Foucault, in so far as the credit for designing the apparatus to be described is concerned.

A rectangular aperture at S (Fig. 121) was illuminated with sunlight, and the pencil issuing from this aperture traversed an achromatic lens L, and was in consequence brought to a focus at a distant point M after reflection from a plane mirror R. The point M lay on a concave mirror, the centre of curvature of which was at the centre of R. Now, if R is rotated about an axis perpendicular to the paper, so long as any light falls on the concave mirror M, the central ray of the convergent pencil will always be reflected at normal incidence; *consequently the reflected pencil will diverge from the point of incidence, and will coincide exactly with the incident convergent pencil.* After once more falling on R, the light will be reflected back to L, and will finally be brought to a focus at S.

[1] *Sur la Vitesse de la Lumière*, by A. Cornu. *Rapports présentés au Congrès international de Physique*, Paris, 1900, tome ii., p. 232.

If, as in Fig. 121, a plane parallel plate of glass is interposed between L and S, part of the returning light will be reflected aside, so as to form an image at *a*. It follows that if R is rotated slowly, the image at *a* will disappear when the light ceases to fall on M ; *but while light falls on M, the image at "a" will be perfectly stationary.*

Now let us suppose that the speed of rotation of R is increased. Light will occupy an appreciable interval of time in travelling from R to M and back again, so that in the interval between its successive reflections from the mirror R, the latter

FIG. 121.—Fizeau and Foucault's Experiment.

will have had time to rotate through an appreciable angle. As a consequence, the returning light will traverse a new path after reflection at R, and if the mirror R rotates in the direction in which the hands of a clock revolve, the returning light will finally form an image at S′, the image obtained by reflection from the glass plate being at the same time deflected to *a′*. From a measurement of the distance *a a′*, we can obtain the angle through which the mirror R has rotated while light travelled from R to M and back. If the speed of rotation of the mirror is determined, the time required for the latter to rotate through this angle is known. Thus, the velocity of light becomes known in terms of the deflection *a a′* of the image seen, the speed of rotation of the mirror R, and the distance RM.

In Foucault's experiment, the distance RM was equal to 20 metres. The mirror was rotated by a species of air turbine such as has since

been used in most cases where a small mirror has been required to rotate at a high speed. The deflection, *a a′*, of the image only amounted to 0·7 mm., but this small deflection was partially compensated for by the circumstance that, owing to the perfection of the design of the optical arrangements, a perfect image of the aperture was obtained. As a consequence, a vertical cross-wire stretched across the luminous aperture at S was clearly reproduced in the image, and the motion of this was measured by means of a micrometer eye-piece. The cross-wires of the micrometer could be set to within 0·005 mm., so that the error of measurement amounted to 1/300, or at most to 1/150. Thus, Foucault's results were not of the highest degree of accuracy. His value for the velocity of light was 298,000,000 metres per second.

The most important advance effected by Foucault was his proof that **light travels more slowly in water than in air.** He placed a long tube filled with water between R and M (Fig. 121), and found that the deflection *a a′* of the image was thereby increased. Thus, the introduction of the tube of water had a similar effect to increasing the distance RM.

**Criticism of the Fizeau-Foucault Method.**—M. Cornu has pointed out that the trustworthiness of the value of the velocity of light, as determined by the aid of a rotating mirror, depends on the truth of two assumptions, which imply an intimate knowledge of the nature of light.

These assumptions are :—

(1) The laws of oblique reflection, from a mirror moving with a velocity small in comparison with the velocity of light, are the same as for a stationary mirror.

(2) The laws of reflection of rays, forming a real image which itself moves transversely to the direction of the rays, with a speed more or less comparable with that of light, are the same as if the image were stationary.

With regard to (2), it will be remembered that the image formed on the surface of M is moving across that mirror with a very great velocity during the period when reflection occurs.

**Disadvantages of the Fizeau-Foucault Method.**—An important drawback to the Fizeau-Foucault method of determining the velocity of light lies in the circumstance that if, in order to increase the deflection of the resultant image, we increase the distance RM (Fig. 121), the brightness of the image is corre-

spondingly diminished. This is evident, if we notice that during a quarter revolution of R, the image which sweeps across M travels round a semicircle through a distance equal to $\pi r$, where $r =$ RM. Thus, light will only be returned from M during a small fraction of the time required for a rotation of R. This fraction has the value $a/4\pi r$, where $a =$ the breadth of the mirror M. Consequently, the brightness of the image when R is rotating, is to its brightness when R is stationary, as $a$ is to $4\pi r$. If $r$ is increased, the brightness of the image formed by reflection from the rotating mirror is obviously diminished.

There is no corresponding disadvantage in the employment of Fizeau's rotating wheel method. In this case an increase in the distance through which the light travels makes no difference in the brightness of the image, except in so far as the light is absorbed by the atmosphere. M. Cornu increased the distance FR (Fig. 120) from 8,633 metres (the distance used by Fizeau) to 22,910 metres. Experiments are in course of execution at the Nice observatory with FR equal to 40,000 metres (nearly 25 miles).

**Michelson's Method.**—Professor Michelson has effected a re-determination of the velocity of light by the aid of a rotating mirror; by an ingenious modification of the Fizeau-Foucault optical system, he has been able to increase the distance RM (Fig. 121) to 600 metres, the deflection of the resulting image amounting to 133 mm. The most important alteration introduced was the transference of the lens L, from between S and R, to a position between R and M. In this way the brightness of the final image is rendered almost independent of the distance RM.

The advantage of this modification can be explained by reference to Fig. 122. Two positions of the rotating mirror R are there shown; the construction for the images $I_1$ and $I_2$, due to reflections of S in R when in these positions, is similar to that explained on p. 25. Let L be the position of the lens between R and M, and let $f$ be its first principal focus. Then, if the lines $I_1f$ and $I_2f$ produced cut the rotating mirror and the lens L, they will represent rays diverging from $f$, and will consequently, after refraction through L, be reduced to parallelism with the axis of L. The images formed by this refraction are obtained by drawing straight lines from $I_1$ and $I_2$, through the centre of L. Let these images be formed on opposite edges of a concave mirror, M. If the aperture of the lens L is sufficiently wide, it is clear that

the light reflected from the rotating mirror will reach $M_1$, whenever the image of S in R lies between $I_1$ and $I_2$ (Fig. 122).

Now, during a quarter revolution of R, the image of S in R describes a semicircle, with radius RS $= d$ (say). Let the distance $I_1I_2 = \alpha$. Then light will reach M during a fraction of the time of a complete revolution of R, equal to $\alpha/4\pi d$. This fraction will also represent the ratio in which the brightness of the final image is reduced by setting R in rotation.

FIG. 122.—Illustrates the advantage of Michelson's Method.

Let $\beta$ be the breadth of the mirror M. Then it is evident from Fig. 122, that the line $I_1I_2$, equal in length to $\alpha$, forms, by refraction through L, an image of length equal to $\beta$. Consequently $\beta/\alpha$ gives the magnification of the line $I_1I_2$ by the lens L. Also, if the first focal distance of L is equal to $f_1$, while the distance between $I_1$ or $I_2$ and L is equal to $u$, we have (p. 72)—

$$\frac{\beta}{\alpha} = \frac{f_1}{u - f_1}; \quad \therefore \alpha = \beta \cdot \frac{u - f_1}{f_1}.$$

Therefore, the ratio in which the brightness of the final image is reduced by setting R in rotation, is equal to—

$$\frac{u - f_1}{f_1} \cdot \frac{\beta}{4\pi d}.$$

Consequently, keeping $\beta$ and $d$ constant, the brightness of the final image will remain unaltered if we increase $u$ and $f_1$ in the same proportion, so as to keep $(u - f_1)/f_1$ constant. But, if we increase $u$ and $f_1$ in any ratio, $v$, (which will be equal in this case to the distance LM) will be increased in the same ratio. Thus, by using a lens of sufficiently long focus and a sufficiently wide aperture, placed at a suitable distance from R, we can increase the distance RM as much as we please without diminishing the brightness of the final image.

An increase in $d$ will obviously diminish the brightness of the final image; on the other hand, with a given speed of rotation of the mirror R, and a given distance RM, the line SS′ (Fig. 121) will subtend a constant angle at R, so that the *distance* between S and S′ will increase with an increase in the value of $d$ (= RS).

In Michelson's experiment, the source S was a narrow vertical slit illuminated by sunlight. The lens L (Fig. 122) had a focal length of 150 feet and was placed between R and M, at a distance of about 135 feet from R. The distance between S and R was 30 feet, while that between R and M was 2,000 feet. It will be seen from Fig. 122, that, for a ray passing centrally through L to be reflected back along its previous path, M must be a concave mirror with its centre of curvature at the centre of L. This is the condition that the final image shall be an exact reproduction of the source S. In view, however, of the difficulty of correctly shaping a concave mirror with a radius of curvature as great as 1,865 feet, Michelson used a plane mirror 7 inches in breadth, at M. As a consequence, the final image was not an exact reproduction of S, but was broader and slightly undefined. In this case a vertical cross-wire, stretched across the aperture at S, would not be seen in the final image. To compensate for this indefiniteness, however, Michelson obtained a deflective of the final image amounting to 133 mm., instead of 0·7 mm. as obtained by Foucault. The glass plate used by Foucault (Fig. 121) was dispensed with, the deflected image being directly observed by the aid of a telescope placed towards one side of S. The mirror, which was polished only on one side, was driven by an air turbine at a

speed of 256 revolutions per second. Its speed was regulated by throwing a beam of light reflected from it on to a mirror carried by the prong of a vibrating tuning fork, so that the beam was reflected thence into a telescope. When the beam of light fell on the mirror carried by the tuning fork, at intervals exactly equal to the period of vibration of the latter, the spot of light seen in the telescope appeared stationary (p. 177). According to Michelson's determination, the velocity of light lies between 299,793,000 metres per second, and 299,913,000 metres per second.

There was no sign of the observed image being drawn out into a spectrum, as would be the case if the velocity of light varied with the colour of the latter.

**Newcomb's Experiments.**—Newcomb, with the aid of Michelson, has made a further determination on the same general principles. He increased the distance RM to over 12,000 feet, and used a mirror which could be rotated in opposite directions. By this means the final image was deflected first to one side, and then to the other, of S, and the distance to be measured was doubled. The mirror comprised four reflecting surfaces, forming faces of a cube; thus the brightness of the image was further increased fourfold. Newcomb's final result was that the velocity of light lies between 299,830,000 metres per second, and 299,890,000 metres per second.

**Most Probable Value of the Velocity of Light.**—The two determinations of the velocity of light, which make the greatest claims to accuracy, are those of Cornu, and Michelson and Newcomb. The respective values obtained are

| | |
|---|---|
| *According to Cornu* . . . | 300,400,000 ± 300,000 metres per second. |
| *According to Michelson and Newcomb* . . . . | 299,860,000 ± 30,000 ,, ,, |

It will be noticed that Michelson and Newcomb's value lies outside the limits given by Cornu, being slightly smaller. The general agreement is very good, but the question may be raised: does the true velocity of light lie between the comparatively narrow limits given by Michelson and Newcomb? This point has been discussed by M. Cornu.[1] In his opinion, the un-

[1] "Sur la Vitesse de la Lumière," *Rapports présentés au Congrès International*, tome ii., p. 235.

certainty of the truth of the assumptions mentioned on p. 228 must be taken into account in this connection. In answer to this criticism, it may readily be admitted that the accuracy of the rotating mirror method depends on the truth of the assumptions specified; but, on the other hand, the present state of our knowledge of the nature of light appears to justify these assumptions.[1]

**The Corpuscular Theory of the Nature of Light.**—In the speculations of the ancients (such as Plato and Aristotle), light was considered to be merely a property of the eye, which virtually had the power of throwing out invisible tentacles, thus becoming cognisant of the nature of distant objects. It is unnecessary to criticise such speculations in detail; the art of photography, and many experimental investigations, have proved that the effects of light may be detected by methods which do not depend on any property of the eye.

According to the corpuscular theory, light consists of a swarm of material particles moving at a great speed; these particles are supposed to be emitted by a luminous body, very much as shots may be fired from a gun. Their mechanical impact on the retina produces the sensation of light. They move in straight lines as long as they continue to travel through interstellar space, just as a projectile would do in similar circumstances. On approaching to within a certain very small distance from the surface of a material medium, the path of the luminous corpuscle is modified.

The nature of this modification varies according as the corpuscle is in a condition favourable to reflection or transmission. In the former case it experiences a repulsion normal to the surface, so long as it remains within a certain very small distance from the latter. If we resolve the velocity of the particle into components respectively perpendicular and parallel to the surface, the repulsion first neutralises and then reverses the perpendicular component, while leaving the other component unaffected. Thus, in the immediate neighbourhood of the surface the path of the corpuscle is curved (Fig. 123). The initial and final paths of the corpuscle are rectilinear, and are equally inclined to the normal to the surface.

1 See Dr. O. Lodge, on Aberration Problems, *Phil. Trans.* vol. 184 (1893) A., p. 740.

If, on approaching the surface, the corpuscle is in a condition favourable to transmission, it experiences an attraction toward the more refracting medium. Let the path of the corpuscle be as represented in Fig. 124, the lower medium being the more refracting; then the component velocity of the corpuscle, perpendicular to the surface, will be increased as it passes through a thin layer bounded by two planes parallel to, and on opposite sides of, the surface, while the component velocity parallel to the surface remains unaffected. After traversing the layer mentioned, the velocity of the corpuscle experiences no further change. Thus, if $i$ and $r$ are the angles which the initial and final paths make with the normal to

FIG. 123.—Reflection of Light, on the Corpuscular Theory.

FIG. 124.—Refraction of Light, on the Corpuscular Theory.

the surface, and if $v$ and $v'$ are the respective velocities in the upper and lower media, then $v \sin i$ will be the component velocity parallel to the surface before refraction, and $v' \sin r$ will be the corresponding value after refraction. Thus,

$$v \sin i = v' \sin r, \text{ and } \sin i/\sin r = v'/v.$$

Thus, the ratio of the sines of the angles of incidence and refraction will be constant, in accordance with Snell's law. But, since the ratio is greater than unity when light is refracted from a rarer to a denser medium, the ratio $v'/v$ must be greater than unity, and *in the denser (more refracting) medium the velocity of light must be greater than in the rarer medium.* Thus the corpuscular theory of light involves an essential

condition which can be tested experimentally. As we have seen, Foucault proved that light travels more slowly in water than in air. From the moment of the completion of Foucault's experiment, the corpuscular theory became untenable.

In order to account for the simultaneous reflection and refraction of light, Newton assumed that at the surface of separation of two media, the luminous corpuscles are subject to "fits" of easy reflection and easy transmission. To explain these fits, he asked "when a ray of light falls on the surface of a pellucid body, and is there refracted or reflected, may not waves of vibrations, or tremors, be thereby excited in the refracting or reflecting medium at the point of incidence and continue to arise there, and to be propagated from thence . . . . and are not these vibrations propagated from the point of incidence to great distances? And do they not overtake the rays of light, and by overtaking them successively, do they not put them into the fits of easy reflexion and easy transmission described above? For if the rays endeavour to recede from the densest part of the vibration, they may be alternately accelerated and retarded by the vibrations overtaking them."[1]

**Fundamental Conditions to be complied with by a Theory of the Nature of Light.**—To account for the properties of light, it must be remembered that, besides exciting the sense of vision in the eye, it may, when absorbed by a body, produce a rise of temperature.[2] Since heat is a form of energy, the propagation of light must be accompanied by a transference of energy. If the luminous corpuscles mentioned above possessed the property of inertia, then, when moving with a definite velocity, they would possess energy. Light reaches us from the stars, so that the existence of a *material* medium through which it may be propagated is not necessary. Light is propagated in approximately straight lines, with a definite velocity, the value of this velocity for propagation in transparent material media being less than in a vacuum. At this point the corpuscular theory breaks down. Our only alternative is to inquire whether light may not be the result of movements transmitted through some non-material medium which pervades all space. If this medium is endowed with the property of inertia, then the transmission of any movements through it will

[1] *Opticks*, fourth edition, 1750, book iii., query 17.
[2] Edser's *Heat for Advanced Students*, p. 437.

be accompanied by the transmission of energy. In framing an hypothesis as to the nature of these movements, analogy with the transmission of sound through a gas suggests that they may be of the nature of vibrations or waves. On the other hand, sound is not transmitted in straight lines; it can reach us when the position from which it originated is hidden from sight, say by an intervening building. This objection was considered by Newton to be fatal to the wave theory of light. However, it is now known that sound waves of very high frequency are propagated with a rough approximation to a rectilinear path. Thus, a hand placed some distance in front of the ear will appreciably screen off a very high note, such as that from a high pitched whistle. Before deciding against the wave theory of light, it consequently becomes necessary to inquire whether waves of excessively high frequency may not be propagated rectilinearly; and further, whether, as a matter of fact, light is propagated, in all circumstances, in perfectly straight lines? These points will engage our attention in the ensuing chapters.

## Questions on Chapter XI

1. Describe and explain the method of determining the velocity of light from observations on Jupiter's satellites.

2. How has the velocity of light in interplanetary space been measured?

3. Explain the phenomenon of the aberration of light, and describe the apparent motions of the stars, which are due to it.

4. Describe Fizeau's method of measuring the velocity of light. How would the appearances seen during the experiment be changed, if light of different colours travelled through air with different velocities?

5. Explain carefully some one experimental way of measuring the velocity of light.

6. The true velocity of light being known from terrestrial experiments, explain how observations of the eclipses of Jupiter's satellites could be used to determine the diameter of the earth's orbit.

# CHAPTER XII

## VIBRATIONS AND WAVES

**Composition and Resolution of Displacements.**—When a body is moved from one position to another, the straight line drawn from the initial to the final position is termed the displacement of the body. It follows that a displacement is a *distance measured in a definite direction;* its description involves a numerical magnitude, together with a specification of the direction of measurement. More generally, any measurement which involves, not only a magnitude, but a direction, is termed a **vector**, or directed quantity.

FIG. 125.—Composition of Displacements.

Let the vectors OA, AB (Fig. 125) indicate, in magnitude and direction, two successive displacements of a body initially at O. Then the resultant displacement of the body is equal to the vector OB. In other words, the two successive displacements $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are equivalent to the single displacement $\overrightarrow{OB}$. Thus, we may write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

We may, however, proceed in a different manner. Let us displace the body from O, first along OA, through the distance $Oa_1$, equal to $OA/n$, where $n$ is any number; and then through $a_1b_1$, which is equal to $AB/n$, and is measured parallel to AB. These two successive displacements are equivalent to the single

displacement $Ob_1$. If we next displace the body through the distance $b_1a_2$, parallel to OA and equal to OA/$n$; and then through the distance $a_2b_2$, parallel to AB and equal to AB/$n$, the body will have arrived at $b_2$. Repeating this procedure until $n$ displacements, each equal to OA/$n$, have been effected parallel to OA, and an equal number of displacements, each equal to AB/$n$, have been effected parallel to AB, it is evident that the body finally arrives at B. Its actual path has been along the zig-zag line $Oa_1\ b_1\ a_2\ b_2\ a_3\ b_3$ . . . . . B, and the resultant displacement is equal to OB. If we imagine that $n$, the number of steps in the zig-zag path, is increased indefinitely, then the latter approximates to the straight line OB. The body still suffers the displacements OA and AB, but an infinitesimal displacement parallel to OA is followed by an infinitesimal displacement parallel to AB and so on. Thus, the conditions are practically the same as if the displacements OA and AB were performed simultaneously, and we see that the result is the single displacement OB.

An instance of the simultaneous superposition of displacements occurs when a person shifts his position in a moving railway carriage. The resultant displacement of the person is equivalent to the displacement due to the motion of the carriage, together with the motion of the person relative to the carriage.

Any two displacements OA and AB are equivalent to a displacement OB; and conversely, any displacement OB can be replaced by the component displacements OA and AB. Any magnitude and direction may be chosen for one of the components, such as OA, but when this is given, the remaining component AB becomes known. If the directions of the components are given, it is easily seen that the magnitude of the components becomes known.

FIG. 126.—Resolution of a Displacement into Rectangular Components.

The most useful method of resolving a displacement is to choose its components so that the angle OAB $= \frac{\pi}{2}$ (Fig. 126).

Let OA $= x$, while AB $= y$ and OB $= r$. Let the angle AOB $= \theta$.
Then, $x/r = \cos\theta$, and $x = r\cos\theta$.
$y/r = \sin\theta$, and $y = r\sin\theta$.

**Composition and Resolution of Velocities.**—When a body is moving uniformly in a straight line, its velocity is equal to the distance through which it moves in one second. If the body is not moving uniformly, its velocity at any instant is equal to the distance it would cover in one second if it continued moving as at the instant under consideration. It follows that a velocity is a displacement per unit time. Thus, velocities may be compounded or resolved in the same manner as displacements.

**Composition and Resolution of Forces.**—When the velocity of a body is variable, the rate of increase of the velocity measures the force acting on each unit of mass of the body, in the direction of the increase of velocity. The unit of force (the dyne) when acting on a gram of matter produces unit (*i.e.* 1 cm. per sec.) increase in the velocity in every second during which it acts.

Let a certain force when acting on a gram of matter increase its velocity by the distance OA (Fig. 127) per second in each second. Then OA represents this force in magnitude and direction. Let any other force be represented in a similar manner by OC. Then, these two forces, when acting simultaneously on a gram of matter, would, during a second, increase its velocity by the components OA and OC. From A draw AB equal and parallel to OC, and join OB. Then the velocity of the gram of matter will be actually increased by the distance OB per second during each second. A single force equal to OB would also produce the same increase of velocity. Thus, OB represents the **resultant** of the forces OA and OC. Consequently forces may be compounded and resolved in the same manner as displacements and velocities.

FIG. 127.—Composition of Forces.

**Periodic Motion.**—When a body moves in such a manner that it periodically retraces its path, the motion is said to be periodic. The time which elapses between successive passages

*in the same direction* through any point is termed the **period** of the motion.

The motion of the hands of a clock is periodic, the period of the motion of the minute hand being one hour, or 3600 seconds. The bob of a pendulum moves periodically, the period being equal to the time of one complete (to and fro) oscillation.

**Simple Harmonic Motion** (S.H.M.).—A particular kind of periodic motion demands special attention In Fig. 128, AB represents a slotted bar rigidly connected to a rod CE, which works in guides at E and D. Thus, the slotted bar AB is only capable of moving at right angles to its length, or parallel to the direction CE. A pin, P, is carried by a circular disc which can be rotated about C as centre; this pin works in the slot of the bar AB. Now, as the disc rotates, the pin P describes a circle; the direction of its motion is thus continually changing. The component of its motion perpendicular to AB will be communicated to the slotted bar and its guiding rod; while the component, parallel to AB, will produce no effect on the motion of the slotted bar. The slotted bar will thus move up and down as the pin P describes its circular path: it is said to perform a *simple harmonic motion* (S.H.M.).

FIG. 128.—Mechanism producing Simple Harmonic Motion.

The characteristic properties of a S.H M. can be studied by the aid of Fig. 129. Let a tracing point, P, revolve uniformly about O as centre, at a constant distance from it, equal to OP; the direction of motion being opposite to that of the hands of a clock. Through O draw the rectangular axes X'OX and Y'OY. From P draw PQ perpendicular to OY. Then, treating OP as a vector, we see that it is equivalent to the components OQ and QP. Thus, OQ is the component of OP resolved parallel to OY. As P revolves about O, the point Q will move up and down along Y'OY. When P passes across the axis X'OX, Q will pass through O. When P passes across the axis

Y′OY, Q will be at its position of maximum displacement relative to its mean position, O. The maximum displacement of Q will thus be equal to the radius of the circular path of P; it is termed the **amplitude** of the S.H.M. The **phase** of the S.H.M. at any instant is equal to the angle which has been swept out by the line OP, measured from some

FIG. 129.—Characteristics of a S.H.M.

fixed line. It is convenient to measure the phase from the particular position of OP when Q is moving in a certain direction (say upwards) through O. Thus, in Fig. 129, the angle XOP $=\theta$, is equal to the phase of the S.H.M. when P has the position indicated.

The displacements of Q for various values of the phase angle $\theta$, are shown by the curve to the right of Fig. 129. The distance $ar$, measured along the horizontal axis, is taken equal to the circular measure of the angle XOP, and the distance $rp$ plotted vertically above $r$ is equal to the corresponding value of OQ. Other points on the curve are obtained in a similar manner.

Let OP $= a$, while OQ $= y$. Then $y/a = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Therefore $y = a\sin\theta$. This is the equation to the curve to the right of Fig. 129.

When the tracing point P crosses the axis OX, it is moving parallel to the axis OY. Consequently, at this instant the point Q is moving along OY with a velocity equal to that of the tracing point P in its circular path. Let the tracing point complete a revolution about O in a time T. Since the length of its circular path is equal to $2\pi a$, the velocity of P is equal to $2\pi a/\mathrm{T}$. Hence, **the velocity of Q as it passes through its mean position O is equal to $2\pi a/\mathrm{T}$.**

When the tracing point P crosses the axis OY, it is moving in a direction perpendicular to OY. At this instant the point Q will be stationary. Consequently, **at the extremity of its excursion on either side of its mean position O, the point Q is for an instant stationary.**

Thus, a point executing a S.H.M. moves with a maximum velocity equal to $2\pi a/T$ on passing through its mean position. Its velocity then diminishes as it recedes from its mean position, and becomes equal to zero at the extremity of an excursion. Subsequently, as the point returns toward its mean position, its velocity increases, and once more attains the value $2\pi a/T$ on moving through the mean position.

**Resolution of a Circular Motion into its Harmonic Constituents.**—The vector OP (Fig. 129) is equivalent to the two components OR $= x$, and RP $= y$, at right angles to each other. Now, OR/OP $= x/a = \cos\theta$, and, therefore, $x = a\cos\theta$.

As P moves uniformly round its circular path, the point R will move backwards and forwards along the axis X′OX, and will obviously execute a S.H.M. R will be at its position of maximum displacement at the instant when Q is passing through O. Consequently the phases of $y$ and $x$ differ by $\pi/2$.

This can also be shown as follows. We have, for the S.H.M executed along X′OX—

$$x = a\cos\theta.$$

For the S.H.M. executed along Y′OY we have—

$$y = a\sin\theta = a\cos\left(\frac{\pi}{2} - \theta\right) = a\cos\left(\theta - \frac{\pi}{2}\right).$$

Therefore the phase of $y$ is behind that of $x$ by $\frac{\pi}{2}$.

Thus, **we can always decompose a uniform circular motion into two S.H.M.'s at right angles to each other, the amplitudes of the latter being equal, but their phases differing by $\pi/2$. Conversely, two S.H.M.'s at right angles to each other, of equal amplitudes but with phases differing by $\pi/2$, can be replaced by a uniform circular motion, the radius of the circle being equal to the amplitude of either S.H.M.**

Let the point P complete one revolution in a time T. If a time $t$ is occupied in describing the arc XP, we have $\theta = 2\pi t/T$. Thus, for the equations of the mutually rectangular S.H.M.'s into which the circular motion of P may be decomposed, we have—

$$x = a\cos(2\pi t/T);$$
$$y = a\sin(2\pi t/T).$$

T is the period of the S.H.M.'s, or of the equivalent circular motion. $a$ is the amplitude of either S.H.M.

We have heretofore supposed that P revolves about O in a direction opposite to that in which the hands of a clock revolve. If we now suppose that it moves in the same direction as that in which the hands of a clock revolve, $\theta$ will be negative, and we must put $\theta = -2\pi t/T$.

The equations of the corresponding S.H.M.'s will be

$$x = a\cos\theta = a\cos(-2\pi t/T) = a\cos(2\pi t/T):$$
$$y = a\sin\theta = a\sin(-2\pi t/T) = -a\sin(2\pi t/T).$$

**Superposition of Two Equal and Opposite Circular Motions.**—Let two tracing points start simultaneously from A (Fig. 130), and move round the circle ABCD in equal times, but in opposite directions. In each revolution these points will pass each other at A and C. Let us call the circular motion executed in the direction in which the hands of a clock revolve, a right-handed circular motion; that executed in the opposite direction being termed left-handed. Then we can decompose each circular motion into its harmonic constituents, so that we have—

FIG. 130.—Composition of Two Opposite Circular Motions.

$$\left.\begin{aligned} x_1 &= a\cos(2\pi t/T) \\ y_1 &= -a\sin(2\pi t/T) \end{aligned}\right\}\text{Right-handed circular motion.}$$
$$\left.\begin{aligned} x_2 &= a\cos(2\pi t/T) \\ y_2 &= a\sin(2\pi t/T) \end{aligned}\right\}\text{Left-handed circular motion.}$$

If we communicate these two circular motions simultaneously to a body, at any time $t$ its component displacement parallel to the axis of $x$ will be equal to $x_1 + x_2$, or $2a\cos 2\pi t/T$. Its

component displacement parallel to the axis of $y$ will be equal to $y_1 + y_2$, which is equal to zero. Thus, **two circular motions, starting simultaneously from A, and executed in equal times but in opposite directions, are equivalent to a single S.H.M. executed in the line CA.**

It is obvious that if the tracing points pass each other at the points A′ and C′ (Fig. 130), then the resultant S.H.M. will be executed in the line C′A′.

Let us now suppose that the tracing points start from A in opposite directions, but complete their circular paths in times which are not exactly equal. Let the left-handed circular motion be executed in less time than the right-handed one. Then, if the difference of the periods is very small, the two tracing points will, in their first revolution, pass each other at a point very near to C, and again at a point very near to A. This will correspond to a S.H.M. along the line AC. But after a number of revolutions the tracing point moving in the left-handed direction will reach C before the arrival of the tracing point moving in the opposite direction. Consequently the two tracing points will now pass each other at a point C′ between C and D, and again at A′, between A and B. This will correspond to a S.H.M. executed in the line C′A′. As time elapses, the direction C′A′ of the resultant S.H.M. will be rotated through a greater angle from CA.

Thus, **two uniform circular motions executed in opposite directions, and in periods which are not exactly equal, are equivalent to a single S.H.M. executed in a straight line which slowly rotates in the direction of the quicker circular motion.**

**Composition of any two S.H.M.'s executed at right angles to each other, in equal Periods.**—A graphical solution of this problem is given in Fig. 131. Let the amplitude of the S.H.M. executed along the line X′OX be equal to the radius of the outer circle, while that of the S.H.M. executed along the perpendicular axis Y′OY is equal to the radius of the inner circle. Since the periods of the two S.H.M.'s are equal, the respective tracing points will traverse these circles in equal times, in a direction opposite to that in which the hands of a clock revolve. The positions of the tracing points at any particular instant will depend on the initial phases of the S.H.M.'s. If the phases were initially equal, the tracing point on the outer

circle passes through the line OX when that on the inner circle passes through OY. Fig. 131 is constructed on the supposition that the phase of the S.H.M. executed parallel to Y′OY is in advance of that executed parallel to X′OX by $\pi/4$. Consequently, when the tracing point on the outer circle passes through the point marked 1 on that circle, the other tracing point will pass through the point marked 1 on the inner circle. Starting from these points, divide the circumferences of the two circles into the same number of equal parts, and number the dividing

FIG. 131.—Composition of Two Rectangular S.H.M.'s.

points consecutively in the direction of motion. Then, the tracing points on the two circles will pass simultaneously through the points bearing similar numbers. The component displacements at any instant are found by drawing perpendiculars from corresponding points, on the outer and inner circles, to the lines OX and OY respectively. The resultant displacement at that instant is found by compounding these components in the manner explained on p. 237. Thus, a straight line drawn from O to A will give the resultant displacement at the instant when the tracing points pass through the points marked 1.

The construction for the resultant displacements at other instants will be seen on inspection of Fig. 131. It thence becomes evident that the two S.H.M.'s of unequal amplitudes and phases are together equivalent to an elliptic motion. The solution of the following particular cases can be effected on similar lines, and may be left as exercises to the student.

1. When the amplitudes are unequal, and the phases differ by $\pi/2$ or $3\pi/2$, the axes of the ellipse coincide with the axes of reference X'OX and Y'OY.

2. When the amplitudes are equal or unequal, and the phases differ by 0, or $\pi$, the ellipse degenerates into a straight line.

3. When the amplitudes are equal, and the phases differ by $\pi/2$ or $3\pi/2$, the ellipse degenerates into a circle (compare with p. 242).

**Analytical Solution.**—The same problem may be solved analytically as follows:—

Let the amplitudes of the S.H.M.'s along the axes X'OX and Y'OY be respectively equal to $a$ and $b$. At a given instant let the tracing point on the outer circle be in advance of its starting point, on the axis OX, by a distance subtending an angle $\theta$ at the centre O. Then, if $x$ is the component displacement parallel to the axis OX, we have (p. 242)—

$$x = a \cos \theta.$$

This is the equation to the S.H.M. executed along the axis X'OX. If the phase of the S.H.M. executed along the axis Y'OY is in advance of that executed along X'OX by the angle $\delta$, the tracing point on the inner circle will be in advance of its starting point on OY by a distance subtending an angle $(\theta + \delta)$ at the origin. Thus, if $y$ is the component displacement parallel to the axis OY, we have—

$$y = b \cos (\theta + \delta).$$

This is the equation to the S.H.M. executed along the axis Y'OY.

Now— $$\frac{y}{b} = \cos(\theta + \delta) = \cos\theta \cos\delta - \sin\theta \sin\delta$$

$$= \cos\theta \cos\delta - (1 - \cos^2\theta)^{\frac{1}{2}} \sin\delta = \frac{x}{a}\cos\delta - \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \sin\delta.$$

$$\therefore \left(\frac{y}{b} - \frac{x}{a}\cos\delta\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta.$$

$$\therefore \frac{y^2}{b^2} - 2\frac{xy}{ab}\cos\delta + \frac{x^2}{a^2}\cos^2\delta = \sin^2\delta - \frac{x^2}{a^2}\sin^2\delta.$$

$$\frac{x^2}{a^2}\left(\cos^2\delta + \sin^2\delta\right) - 2\frac{xy}{ab}\cos\delta + \frac{y^2}{b^2} = \sin^2\delta.$$

$$\therefore \quad \frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\delta + \frac{y^2}{b^2} = \sin^2\delta.$$

This is the general equation to an ellipse. When $\delta = 0$, $\sin\delta = 0$, and $\cos\delta = +1$. In this case the equation reduces to—

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} + \frac{y^2}{b^2} = \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0.$$

$$\therefore \quad y = \frac{b}{a}x.$$

This is the equation to a straight line passing through the origin, and inclined to the axis of $x$ at an angle of which the tangent is equal to $b/a$. If $\delta = \pi$, $\cos\delta = -1$, and we find that—

$$y = -\frac{b}{a}x.$$

This is the equation to a straight line passing through the origin, and inclined to the axis of $x$ at an angle of which the tangent is equal to $-b/a$. When $\delta = \pi/2$ or $3\pi/2$, $\cos\delta = 0$, and $\sin^2\delta = 1$. In this case we find that—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the equation to an ellipse of which the semi-axes are respectively equal to $a$ and $b$, and coincide with the axes of $x$ and $y$.

When $a = b$, the ellipse degenerates into a circle.

EXPT. 53.—Obtain a piece of clock-spring 5 or 6 inches in length, heat a small portion near the middle point to redness in a blowpipe flame, and twist it while in the flame so that the two halves lie in mutually perpendicular planes (Fig. 132). Sharpen one end, and fix a silvered bead to it. Clamp the opposite end of the spring in a vice. Then the two halves of the spring are capable of bending in directions lying in mutually perpendicular planes. By adjusting the point at which the spring is clamped, the periods of vibration in these directions can be brought into approximate agreement. On setting the free end of the spring in vibration, the bead will be seen to describe a straight line,

FIG. 132.—Spring used in Experiment 53.

ellipse, or circle. If, as generally happens, the periods of vibration in perpendicular directions are not exactly equal, the figure described by the bead will pass from a straight line to an ellipse and thence to a circle, which in its turn will lengthen out into an ellipse and a straight line, and so on.

**Composition of any Number of S.H.M.'s of Equal Periods, executed in the Same Direction.**—Let $OP_1$, $OP_2$ (Fig. 133), be equal to the amplitudes, while the angle $P_1OP_2$ represents the

FIG. 133.—Composition of Two S.H.M.'s of Equal Periods, executed in the same Straight Line.

phase difference, of two S.H.M.'s executed along X'OX in equal periods. Let the points $P_1$ and $P_2$ revolve uniformly in a direction opposite to that of the motion of the hands of a clock, round circles of radii equal to $OP_1$ and $OP_2$, in such a manner that the angle $P_1OP_2$ remains constant. Then OB and OC, the components of $OP_1$ and $OP_2$ resolved parallel to OX, will be equal to simultaneous values of the displacements of the S.H.M.'s. Since these displacements are in the same straight line, their resultant will be equal to OB + OC. From $P_1$ draw $P_1R$ parallel and equal to $OP_2$, and join OR. From R draw

RD perpendicular to OX. Then it is obvious that BD = OC, so that the resultant displacement is equal to OD. Thus, if we suppose the line OR to rotate about O, in a time equal to the period of the component S.H.M.'s, the component of OR, resolved parallel to OX, will give the resultant displacement at any instant. Thus, OR, is equal to the amplitude of the resultant S.H.M., and the phase of the same will be measured by the angle ROX. Thus, **we can compound the amplitudes of S.H.M.'s executed in the same straight line, according to the ordinary laws applying to vectors** (p. 237).

Let the vectors OA, OB, OC, OD, OE (Fig. 134), represent the amplitudes and phases of a number of S.H.M.'s executed in equal periods along the line OX. From A draw A*b* equal and parallel to OB; then the vector O*b* represents amplitude and phase of the resultant of OA and OB. From *b* draw *bc* equal and parallel to OC. Then the vector O*c* represents the amplitude and phase of the resultant of O*b* and OC, that is, the resultant of OA, OB, and OC. Finally, if we draw *cd* and *de* parallel and equal to OD and OE respectively, the vector O*e* will represent the amplitude and phase of the resultant of OA, OB, OC, OD, and OE.

FIG. 134.—Composition of a Number of S.H.M.'s.

**Mechanical Conditions for the Execution of a Uniform Circular Motion.**—Let a body of mass $m$ revolve, with a uniform velocity $v$, round a circle of radius OA = $r$ (Fig. 135). The velocity of the body at any instant will be tangential to the circle, and thus is continually changing in direction, though its magnitude remains constant. Consequently, a force must act on the body.

Let the body move from A to C in a time $t$; then its velocity in this time changes from AB to CD, where the magnitudes of AB and CD are both equal to $v$, while their directions are at right angles to the radii OA and OC respectively. From A draw AE equal and parallel to CD. Then to change AB into AE, we must add to it the vector BE. But

the change of velocity per second is equal to the force acting on each unit of mass of the body. Thus, BE/$t$ is equal to the force per unit mass acting on the body. As $t$ is diminished indefinitely, the lines AB and AE become more and more nearly parallel, and BE becomes more and more nearly perpendicular to AB, or parallel to AO. Thus, **the force acting on the body is directed toward the centre of the circular path.**

FIG. 135.—Mechanical Conditions of a Circular Motion.

The magnitude of the force will be equal to $m$BE/$t$, when $t$ is diminished indefinitely. To evaluate this ratio, notice that the triangles AOC and BAE are similar, both being isosceles, while ∠ AOC = ∠ BAE. Also, in the limit, the arc AC and its chord will be equal; since the body traverses the arc AC in $t$ seconds with a velocity $v$, the arc AC will be equal to $vt$. Then, from the similarity of the triangles AOC and BAE—

$$\frac{\mathrm{BE}}{\mathrm{AB}} = \frac{\mathrm{AC}}{\mathrm{OA}}, \quad \therefore \quad \frac{\mathrm{BE}}{v} = \frac{vt}{r}.$$

Consequently, force acting along AO $= f = m\dfrac{\mathrm{BE}}{t} = m \cdot \dfrac{v^2}{r}$.

This is the well-known expression for the **centrifugal force** acting on a body moving round a circle of radius $r$ with velocity $v$.

Let the body execute $n$ complete revolutions per second. Then, in one second the body moves through a distance equal to $2\pi nr$; this is consequently the velocity of the body. Thus—

$$f = m\frac{(2\pi nr)^2}{r} = mr(2\pi n)^2.$$

If T is the period of one revolution, then $n = 1/\mathrm{T}$. Thus—

$$f = mr\left(\frac{2\pi}{\mathrm{T}}\right)^2.$$

**Mechanical Conditions for the Execution of a S.H.M.**—When a body of mass $m$ executes a uniform circular motion in a period T, it must be acted upon by a force, directed toward

the centre of the circular path, and equal in magnitude to $mr(2\pi/T)^2$, where $r$ is the radius of the circular path. But a uniform circular motion may be decomposed into two S.H.M.'s executed at right angles to each other in periods equal to that of the circular motion.

The vector distance, $r$, from the centre of the circular path to the position of the body at any instant, may be resolved into the rectangular components—

$$x = r \cos \theta, \text{ and } y = r \sin \theta.$$

The central force may be resolved in the same directions, and is equivalent to the rectangular components—

$$m\left(\frac{2\pi}{T}\right)^2 r \cos \theta = m\left(\frac{2\pi}{T}\right)^2 x,$$

parallel to the axis of $x$, and

$$m\left(\frac{2\pi}{T}\right)^2 r \sin \theta = m\left(\frac{2\pi}{T}\right)^2 y,$$

parallel to the axis of $y$, both being directed toward the origin.

Now the motion of a body in a straight line can be affected only by forces acting in that straight line. Thus, the force necessary for the execution of the S.H.M. along the axis of $x$, is equal to $m\left(\frac{2\pi}{T}\right)^2 x$. This force is directed toward the origin, and is proportional to $x$, the displacement from that point.

Similarly, the S.H.M. executed along the axis of $y$ is controlled by a force equal to $m\left(\frac{2\pi}{T}\right)^2 y$, acting along the axis of $y$, and directed toward the origin.

Thus, **a body moving in a straight line will execute a S.H.M. if it is acted on by a force, directed toward a point in the line of motion, and proportional to the displacement of the body from that point.**

**Applications.**—1. *Heavy body suspended by an elastic filament.*—Let A (Fig. 136) represent the equilibrium position of a heavy body suspended by an elastic filament OA. Let $m$ be the mass of the body. Then the downward pull on the body is equal to $mg$, where $g$ is the force per unit mass exerted by gravity. At A this force is just counterbalanced by the tension of the stretched elastic filament. If the body is displaced downwards to B, the increase in the tension of the filament is proportional to the displacement AB; consequently, the resultant force on the body, when at B, is equal to $f \times$ AB, where $f$ is the force of restitution called into play by unit displacement from A. This

resultant force acts toward A, the position of equilibrium of the body.

If the body is displaced upwards to C, the tension of the filament will be diminished by $f \times$ AC, so that the upward pull of the filament is now less than the downward pull of gravity. The resultant force acting on the body at C will thus be equal to $f \times$ AC, directed toward A.

Consequently, when the body is displaced in the line OA through a distance equal to $a$, and then given its freedom, it will be acted on at each instant by a force directed toward A, and proportional to the displacement from that point. It will therefore execute a S.H.M. about the position of equilibrium A. The amplitude of the S.H.M. will be equal to $a$, and the force acting on the body at the limit of an excursion will be equal to $fa$. But it was proved on p. 251 that when a body of mass $m$ executes a S.H.M. of period T, the force corresponding to a displacement $x$ from its position of equilibrium, is equal to $m\left(\frac{2\pi}{T}\right)^2 x$. Thus, when $x = a$, we have—

FIG. 136.—Heavy Body suspended by Elastic Filament.

$$fa = m\left(\frac{2\pi}{T}\right)^2 a\,; \therefore T = 2\pi\sqrt{\frac{m}{f}}.$$

This formula gives us the period T of the S.H.M., in terms of $f$, the restoring force corresponding to unit displacement, and $m$, the mass of the body. The value of T is independent of the amplitude $a$.

We may obtain the same result in another manner. As the body is displaced from its position of equilibrium, work is performed, and the potential energy of the body is increased. At the limit of an excursion the body is for an instant stationary; it then possesses only potential energy. To find the latter, notice that the restoring force varies from zero at A, to $fa$ at B or C. The average restoring force over the path from A to B or C (Fig. 136) is equal to $fa/2$; thus, the potential energy at B or C, which is equal to the work done during the displacement from A to B, or from A to C, will have the value $f\frac{a}{2} \times a = f\frac{a^2}{2}$.

When the body passes through its position of equilibrium the restoring force vanishes, and the body then only possesses kinetic energy. Its velocity at that instant is equal to $2\pi a/T$ (p. 241), and the kinetic energy of the body is consequently equal to $\frac{1}{2}m(2\pi a/T)^2$. Now, if no

energy is lost during the passage from B to A, the potential energy at B must be equal to the kinetic energy at A, by the law of Conservation of Energy. Thus—

$$f\frac{a^2}{2} = \frac{1}{2}m\left(\frac{2\pi a}{\mathrm{T}}\right)^2; \quad \therefore \quad \mathrm{T} = 2\pi\sqrt{\frac{m}{f}},$$

as before.

It should be noticed that the performance of a S.H.M. is characterised by the continual transference of energy from the potential to the kinetic form, and back again to the potential form. At all points in an excursion, the total energy of the body remains constant.

If any frictional forces act on the body, the total energy of the body decreases continually. Thus, its kinetic energy on passing through its position of equilibrium is less than its potential energy at the extremity of the preceding excursion, and the amplitude continually diminishes, and the body finally comes to rest.

2. *Free Vibration of a Simple Pendulum.*—Let a small heavy body, of mass $m$, be suspended from a point O (Fig. 137) by means of an inextensible, massless filament, OA. If the body is displaced to B, and then given its freedom, it will oscillate about the position of equilibrium A, between the limiting positions B and C. Join B and C by the straight line BC, cutting OA in D. When the body is at B, the forces acting on it are (1) $mg$, acting vertically downwards, and (2) a certain tension in the filament OB. These forces are respectively parallel to OD and OB. When the arc of vibration is small, the body moves to and fro practically along the line BC; thus, the resultant force on the body at any instant must act along the line BC. Then, if F is the resultant force acting on the body at B, we have—

FIG. 137.—Free Vibration of a Simple Pendulum.

$$\frac{\mathrm{F}}{mg} = \frac{\mathrm{BD}}{\mathrm{OD}}.$$

When the arc through which the pendulum swings is very small, the point D will practically coincide with A, and the force F will act toward A, the equilibrium position of the body. Also, in that case, OD will be approximately equal to OA, or to $l$, the length of the

simple pendulum. Let DB = $a$, so that $a$ is the amplitude of the vibration. Then—

$$F = \frac{mg}{l}a.$$

Thus, the restoring force is proportional to the displacement, and the pendulum bob will execute a S.H.M.

Employing reasoning similar to that previously used, we have—

$$m\left(\frac{2\pi}{T}\right)^2 a = \frac{mg}{l}a\,;\ \therefore T = 2\pi\sqrt{\frac{l}{g}}.$$

**Forced Vibrations.**—In the foregoing investigations, a body was supposed to be displaced, and then left free to vibrate under the action of the force called into play by its displacement. Such vibrations are said to be *free*. There is another important class of vibrations performed under the

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FIG. 138.—Forced Vibrations of a Simple Pendulum.

periodically. In such a case the body is constrained to vibrate in a period equal to that of the periodic force, and the resulting vibrations are said to be *forced*.

The general nature of forced vibrations may be studied by the aid of Fig. 138; this refers to a pendulum attached to a point of support which itself executes a S.H.M. On starting, the motion of the pendulum is apparently irregular, due to the simultaneous execution of free and forced vibrations. But after a time the free vibrations die down, and the pendulum executes a S.H.M. in a period equal to that of the point of support. Let A'A be the actual length of the simple pendulum, equal to $l_1$.

Then if $T_1$ is the period of its free vibration, $T_1 = 2\pi(l_1/g)^{\frac{1}{2}}$. Let T be the period of the S.H.M. executed by the point of support. A pendulum of length $l$ would complete an oscillation in T seconds, if $T = 2\pi(l/g)^{\frac{1}{2}}$. Thus, the pendulum must vibrate as if it were supported from a point O, by means of a fibre of length $l$. Consequently, if $T > T_1$, we must have $l > l_1$, and the point O will be on the side of A′ remote from A. In this case the phases of the point of support and the pendulum bob will be equal. As the point of support moves from A′ to B′, the pendulum bob moves from A to B (I, Fig. 138). If $T < T_1$ we must have $l < l_1$, and the point O lies between A′ and A. In this case the phases of the point of support and the pendulum bob differ by $\pi$. As the point of support moves from A′ to B′, the pendulum bob moves from A to C; at any instant the point of support and the pendulum bob will be moving in opposite directions (III, Fig. 138).

With a given amplitude of the point of support, it is obvious that the arc AB described by the pendulum bob will increase as the point O approaches A′. When O is situated at A′, $l = l_1$, and the period of vibration of the point of support is equal to the free period of the pendulum. In this case an infinitesimal periodic displacement of the point of support will produce an indefinitely great amplitude of swing in the pendulum bob (II, Fig. 138). This is an instance of the *sympathetic communication of vibrations.*

Lastly, when T is excessively small in comparison with $T_1$, $l$ will be very small in comparison with $l_1$, and the point O will approximately coincide with A. The pendulum bob will then be unable to move appreciably during the time required for the point of support to complete a vibration. Consequently, in this case, the pendulum bob will remain stationary (IV, Fig. 138).

Let $\alpha$ and $a$ be the amplitudes of the S.H.M.'s respectively executed by the pendulum bob and the point of support (Fig. 138, I). Draw BD perpendicular to A′A. Then DB $= \alpha$ and A′B′ $= a$. From the similar triangles BDO and B′A′O, we have—

$$\frac{DB}{OD} = \frac{A'B'}{OA'}, \text{ or } \frac{\alpha}{l} = \frac{a}{l - l_1};$$

$$\therefore \alpha = a\frac{l}{l - l_1}.$$

Also—
$$l = \frac{g}{(2\pi)^2}T^2, \text{ and } l_1 = \frac{g}{(2\pi)^2}T_1^2;$$
$$\therefore \alpha = a\frac{T^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When $T > T_1$, $\alpha$ has the same sign as $a$. As T approaches the value $T_1$, the value of $\alpha$ increases, and becomes equal to infinity when $T = T_1$. When $T < T_1$, $\alpha$ and $a$ have opposite signs, indicating a difference of $\pi$ in the phases of the corresponding vibrations. Finally, when $T = 0$, $\alpha = 0$.

From B′ draw B′E perpendicular to A′B′, cutting DB in E. Then $EB = (\alpha - a)$. As the point of support is displaced from A′ to B′, the position of equilibrium of the pendulum bob is displaced from A to E; we may term A and E its positions of *absolute* and *relative equilibrium*. Similarly, we may term EB, or $(\alpha - a)$, the *relative displacement* of the pendulum bob. From reasoning similar to that employed on p. 253, it follows that the restoring force acting on the pendulum bob when at B is proportional to EB, or $(\alpha - a)$. If $f$ is equal to the restoring force called into play by unit displacement of the pendulum bob when the point of support is fixed at A′, then $f(\alpha - a)$ will be equal to the restoring force when the bob is at B and the point of support is displaced to B′. Let $\mathbf{F} = f(\alpha - a)$. When $\mathbf{F}$ is positive, it will act from B toward D. The tension of the suspending filament will exert a reaction equal to $\mathbf{F}$ on the point of support; when $\mathbf{F}$ is positive, this reaction will tend to increase the displacement of the latter.

From the similar triangles BEB′ and B′A′O, we have—
$$\frac{EB}{B'B} = \frac{A'B'}{OB'}, \text{ or } \frac{\alpha - a}{l_1} = \frac{a}{l - l_1};$$
$$\therefore \alpha - a = a\frac{l_1}{l - l_1} = a\frac{T_1^2}{T^2 - T_1^2}.$$

Consequently—
$$\mathbf{F} = f(\alpha - a) = fa\frac{T_1^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad (2)$$

The reaction due to the pendulum increases from zero to $\mathbf{F}$ as the pendulum bob moves from A to B. It tends to increase or decrease the displacement of the point of support, according as T is greater or less than $T_1$. As the value of T approximates to $T_1$, the value of $\mathbf{F}$ approaches $\pm\infty$. The work, W, performed by the pendulum bob on the point of support, during the displacement of the latter from A′ to B′, is equal to the average value of the reaction (*i.e.* $\mathbf{F}/2$) multiplied by the distance A′B′. Thus—
$$W = \frac{fa^2}{2}\frac{T_1^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $T > T_1$, part of the kinetic energy possessed by the pendulum bob when it passes through its position of equilibrium, A, is afterwards used up, during the excursion from A to B, in producing an increased displacement of the point of support. When $T < T_1$, the point of support does work on the pendulum, the energy of the latter being greater at the extremity of a vibration than when passing through the point A.

Equations (1), (2), and (3) above comprise no magnitudes relating merely to a pendulum. They will apply equally well to any body attracted toward a point with a force proportional to the displacement, the point itself being constrained to execute a S.H.M.

**Wave Motion.**—When a number of bodies are united by means of elastic connections, one of them cannot move without disturbing its neighbours. Thus, a displacement of any one body will produce a disturbance which is handed on from body to body till the whole of them have suffered greater or less displacements. If the first body is constrained to move periodically, then the rest will be constrained to follow its motion, so that all will execute periodic motions of the same period. We may consider an elastic medium, such as a jelly, to consist of an infinite number of particles united by elastic connections. A periodic motion, transmitted from particle to particle of such a medium, is said to constitute a wave motion. It must be noticed that a *particle* does not move continuously in any direction, but oscillates about its position of equilibrium. On the other hand, a *disturbance* communicated to one particle will, as we have seen, be transmitted through the medium ; in a given medium, a disturbance of a given kind will be transmitted with a definite velocity. On being set in motion a particle acquires kinetic energy ; this energy must have been communicated to it by neighbouring particles, since energy, like matter, is uncreatable and indestructible. Consequently, wave motion is accompanied by the transmission of energy.

**Waves transmitted along a Cord.**—Some of the essential characteristics of wave motion may be explained by reference to Fig. 139. AB represents a stretched flexible string in its position of equilibrium. If the point A is caused to move upwards, it will drag the neighbouring portion of the string after it. If the string were rigid, all particles would be obliged to move so as to retain their initial relative positions ;

but as the string is flexible, the motion of any part of it will only exert a finite reaction on the neighbouring particles, and a definite time must elapse before these acquire any appreciable velocity. Thus, the displacement imposed on A will travel along the string with a definite velocity, being handed on from particle to particle of it.

Let us suppose that one end, A, of the string is constrained to execute a S.H.M., and let $A_1$ be the limit of its excursion in one direction. During the time required for this excursion, all points of the string between $A_1$ and $D_1$ have been displaced. In another instant $D_1$ will be set in motion by the reaction of the particle behind it, and still later

FIG. 139.—Wave Motion along a Cord.

the particle in front of $D_1$ will in its turn commence to move upwards. The point $A_1$ is for the moment at rest, having reached the limit of its excursion (p. 241); but all particles between $A_1$ and $D_1$ are moving upwards, and thus possess kinetic energy. These particles continue to move upwards as the point A commences its return journey; each one in turn momentarily comes to rest, when it has yielded up its kinetic energy in setting the more advanced particles in motion; it then commences its return journey. Thus, the motion of any particle of the string is not *immediately* influenced by the motion of the end A. When the end of the string regains its initial position, the disturbance has the form $A_2D_2$. The particle $D_2$ is now just on the point of moving upwards. As the end of the string moves to $A_3$, and back again to its position of equilibrium, the disturbance goes through the form $A_3D_3$, and finally acquires the form $A_4D_4$. The point A has now completed one vibration, and is just on the point of moving upwards at the com-

mencement of a second. The particles at $D_4$ and $A_4$ are moving through their positions of equilibrium in the same direction; they are thus in the same phase (p. 241). The particle midway between $A_4$ and $D_4$ is also moving through its position of equilibrium, but the direction of its motion is opposite to that of $A_4$; the phases of this particle and $A_4$ consequently differ by $\pi$.

If we imagine the end A to continue its harmonic motion, while the string itself is supposed to be indefinitely long (so that the effects produced when the disturbance reaches the end of the string may be left out of account), it is plain that the disturbance $A_4D_4$ will travel along the string, and will be succeeded by other similar disturbances as the end A completes successive vibrations. The disturbance $A_4D_4$ is termed a **wave**, and a number of such disturbances following each other are said to constitute a **wave train**. As a wave train passes any particular particle, the latter executes a S.H.M. similar in every respect to that of the point A. **Particles in the path of a wave train, simultaneously moving in the same direction through positions of equal and similar displacement, are said to be in the same phase.**

In passing along a wave train, the distance between the nearest particles which are in the same phase is termed the **wave-length.** Thus, $A_4D_4$ is the wave-length of the disturbance travelling along $A_4B_4$ (Fig. 139). It will easily be seen by reference to Fig. 139, that particles in the path of a wave train, which are separated by half a wave-length, are moving in opposite directions through positions of equal and opposite displacement. Consequently, **particles in the path of a wave train, which are separated by a wave-length, or any whole number of wave-lengths, will be in the same phase, and thus similarly situated in all respects; those separated by half a wave-length, or any odd number of half wave-lengths, will differ in phase by $\pi$, or some odd multiple of $\pi$, so that their displacements as well as their velocities will be equal in magnitude, but opposite in directions.**

In a wave train, points of maximum displacement in one direction are termed the **crests** of the waves, while those of maximum displacement in an opposite direction are termed the **troughs** of the waves. Thus, two crests or two troughs are separated by a distance equal to a wave-length, or some whole number of wave-lengths. The displacement of a particle, at the crest or trough of a wave, from its position of equilibrium, is termed the **amplitude** of the wave.

**Transverse Waves in general.**—It is instructive to study the properties of a wave train from a slightly different point of view. Let ABCD (Fig. 140) represent part of a long paper riband which is drawn through a slit AD, and travels in the direction of the arrow with a uniform velocity, V. Let a pencil point P execute a S.H.M. along

the fixed line Y′OY, about the mean position O, in a period equal to T. It is plain that a wavy line, which may be termed the wave curve, will be inscribed on the moving paper riband. This wave curve will at any instant represent the displacements at all points in the path of a wave train. To determine the motions of the individual particles in the path of the wave train, imagine that a sheet of paper provided with numerous equidistant slots parallel to Y′OY is maintained stationary above the moving riband. Then through each slot a small length of the wave curve will be visible, and this will move harmonically up and down the slot as the riband moves uniformly onward. The distance

FIG. 140.—Transverse Wave Motion.

from the mean line OX to the visible element of the wave curve will give the displacement of the particle at any instant. It is thus clear that each particle executes a S.H.M. which differs only in phase from that executed by P. Let $a$ be the amplitude of the S.H.M. of each particle. Then, since P moves through O with a velocity equal to $2\pi a/T$, **each particle in the path of the wave train will move through its position of equilibrium on the line OX with a velocity equal to** $2\pi a/T$. When P is at the end of an excursion in one direction, and is thus marking the crest or trough of a wave on the moving riband, it will for the moment be stationary; consequently, the wave curve is parallel to the axis OX in the immediate neighbourhood of a crest or trough. The wave curve cuts the axis OX at an angle $\theta$, which may be determined as follows :—When the tracing point $P_1$ (Fig. 140) was at $N_1$, the pencil point P was at N. As the tracing point moved from $N_1$ to $P_1$, the pencil point marked the approximately straight line NP on the moving riband. Then $\tan \theta = \tan$ PNO = OP/ON. Let $t$ be the time required for the tracing point to move from $N_1$ to $P_1$. In this time the paper riband advanced by the distance ON with a uniform

velocity V. Thus ON = V$t$. Also the arc $N_1P_1$ of the circular path of the tracing point is approximately a straight line parallel to OP. Since the velocity of the tracing point is equal to $2\pi a/T$, we have $OP = N_1P_1 = 2\pi at/T$. Hence, finally, $\tan\theta = OP/ON = 2\pi a/VT$. We shall subsequently find this result very useful.

**Connection between Wave-Length and Period.**—The distance NQ (Fig. 140) is equal to one wave-length (p. 259). Let $NQ = \lambda$. Then during the time required for the moving paper riband to travel, with velocity V, through the distance NQ, the point P has completed one vibration. Thus, if T is the period of the S.H.M. executed by P, we have—

$$\lambda = VT.$$

We have up to the present confined our attention to waves of transverse displacements. The wave curve in Fig. 140 may, however, be used to represent the displacements in a wave train of any character. Thus, in explaining sound waves, in which the particles in the path of a wave train execute longitudinal vibrations, the distance, from the axis OX to the wave curve, will give the instantaneous displacement of a particle parallel to the direction of wave propagation.

**Superposition of Waves.**—When two or more wave trains travel in any directions through a point, the resultant displacement of a particle at that point will, at each instant, be equivalent to the sum of the vector displacements due to the various waves. The velocity of a particle will be the resultant of the velocities due to the various waves. If the crest of one wave and the trough of another pass simultaneously through a point, then, at the instant of their passage, the displacement of a particle at that point will be equal to the difference of the amplitudes of the waves; if the waves are of equal period, and travel approximately in the same direction, then the displacement of the particle will be permanently equal to zero if the amplitudes of the waves are equal.

If two wave trains travel with equal velocities in the same direction, the displacement of any particles in their path can be found by adding the vector displacements due to the two sets of waves. Fig. 141 represents the resultant instantaneous displacements at various points in the path of two wave trains, of which the amplitudes are equal while the wave-length of one is half that of the other.

The resultant displacement curve, due to the superposition of any number of wave trains of unequal wave-lengths, can be found in a similar manner. As the number of wave trains increases, the complexity of the resulting curve is also increased.

FIG. 141.—Superposition of Waves.

Conversely, by superimposing wave trains of various amplitudes, phases, and wave-lengths, we could, by trial, produce a resultant curve of any desired shape.

**Resolution of an Arbitrary Disturbance into Harmonic Waves.**—It has been proved by Fourier that any arbitrary disturbance transmitted through space can be resolved into a number of simple harmonic waves of definite amplitudes, phases, and periods. The following instance may make the meaning of this statement clearer. At Black Gang Chine, in the Isle of Wight, the receding waves cause a violent commotion amongst the stones of the beach. The stones strike against one another at purely arbitrary intervals, which, however, follow each other in rapid succession. Each concussion produces a disturbance in

the surrounding air, and the result of the innumerable concussions of different stones is the production of a note of more or less definite pitch.[1] The note is not, however, pure, but consists of a great number of waves of which the periods vary continuously between certain limits; the amplitude of a wave depends upon its period, attaining a maximum value for a particular wave-length, and falling off rapidly for greater or smaller wave-lengths.

This example will prove valuable when we come to study the production of white light.

**Stationary Waves.**—When two similar and equal wave trains travel in opposite directions along a straight line with

FIG. 142.—Formation of Stationary Waves.

equal velocities, the resultant disturbance of the medium takes the form of a number of stationary waves. The method of production of stationary waves, together with their most important properties, can be understood by the aid of Fig. 142. The

[1] "The scream of a maddened beach drawn down by the wave."—TENNYSON.

curves A and B represent the component wave trains travelling in the directions indicated by the horizontal arrows; they are drawn one below the other in order to avoid confusion. At the instant when the waves have the positions represented, there will be no displacement at any point along the line of transmission; a crest of a wave of train A coincides with the trough of a wave of train B, and at every point the displacements due to A and B are equal and opposite. The resultant displacement curve at this instant takes the form of the horizontal straight line, R. It must be noticed, however, that certain points in this straight line are stationary, while others are moving transversely with considerable velocities. Points at the crests and troughs of the component waves are stationary, and the resultant velocity at points, such as $N_1$, $N_2$, and $N_3$, through which a crest and a trough are simultaneously passing, is equal to zero. But the points P, Q, in the component wave trains, which are passing through their positions of equilibrium, will have velocities (p. 241); these are represented in Fig. 142 by the small vertical arrows. It will be seen that the velocities at these points in the two wave trains are in the same direction, so that the velocity of the points $V_1$, $V_2$, $V_3$, and $V_4$, in the resultant wave train, is equal to twice the velocity of a point such as P or Q.

As the component trains travel in the directions indicated, the point L of train A will arrive above $N_1$ at the same instant as the point M of train B. As the displacements of L and M are equal in magnitude but in opposite directions, the displacement at $N_1$ will still be equal to zero. The transverse velocities at L and M are also equal but in opposite directions, so that the velocity of $N_1$ will be zero. In other words, the displacements of the point $N_1$, due to the wave trains A and B, will always differ in phase by $\pi$ (p. 241). Thus it appears that the points $N_1$, $N_2$, $N_3$, . . . . will remain permanently stationary. These points are termed **nodes**.

The point S of train A will arrive above $V_2$ at the same instant as the point T of train B. The displacements of S and T are equal and in the same direction, so that the resultant displacement at $V_2$ will be equal to twice that of S or T. When each wave has travelled through a distance equal to a quarter of a wave-length, the troughs C and D will simultaneously arrive above $V_2$, so that the resultant displacement at that point will then be at its maximum value, equal to twice the amplitude of either of the component waves. At this instant the

resultant displacement curve takes the form of the continuous curved line in R.

When each wave has travelled through a distance equal to half a wave-length, the resultant curve again takes the form of the straight line through R; the original velocities at $V_1$, $V_2$, $V_3$, and $V_4$ will now be reversed. As the component waves travel through another quarter of a wave-length, the resultant displacement changes to the form represented by the broken curved line in R. After moving through yet another quarter of a wave-length, the component waves will constitute a train identical with that with which we started, so that the resultant displacement once more takes the form of a straight line.

The points $V_1$, $V_2$, $V_3$, $V_4$ . . . , which suffer the maximum displacement, are termed **antinodes**; the portions of the displacement curve lying between two nodes is termed a **ventral segment.** The distance between two successive nodes, or two successive antinodes, is equal to half a wave-length. It will be obvious from the above that antinodes are alternately positions of maximum displacement and of maximum velocity. Nodes are points of zero displacement and zero velocity.

Stationary waves may be formed by sending properly timed waves along a rope, one end of which is attached to a fixed support. They may also be formed in a long trough containing water.

**Waves transmitted through the Body of a Medium.**—The ripples produced when a stone is thrown into still water must be familiar to every one. This class of waves can only be produced at the surface of a medium. Waves can, however, be transmitted through the body of a medium. The student is probably familiar with the characteristics of the waves transmitted through a gas, and constituting sound. Each particle of the gas in the path of a wave train is thrown into oscillations parallel to the direction of wave propagation. A gas offers no resistance to a mere change of shape, so that the only waves which can be propagated through it are those involving a change of volume.

The conditions necessary for the production of waves in a medium are :—

1. A small element of the medium, when set in motion, must possess kinetic energy.

2. The relative displacement of an element with respect to its surroundings must produce a reaction tending to restore the element to its initial position. In overcoming these reactions,

work must be done during the displacement, so that the medium must be capable of acquiring potential energy.

For reasons which will be explained later, the class of waves which alone can be appealed to as explaining the properties of light, is characterised by the peculiarity, that the displacements are at right angles to the direction of wave transmission. In this respect such waves resemble those transmitted along a stretched string. But the possibility of transmitting transverse waves along a string is entirely due to the circumstance that the tension of the string gives it a capacity to resist any change of shape. Consequently, we must study the propagation of transverse waves in a medium endowed with a capacity to resist any change of shape. Such a medium will possess elastic properties similar to those of an ordinary jelly; it is termed an **elastic solid.**

It must be remembered that the term "solid" is applied to any substance which can maintain its shape without lateral support; it does not necessarily imply hardness or impenetrability. Thus, as Maxwell pointed out, sealing-wax is really a viscous fluid, since if supported at its ends, its shape slowly but progressively changes; while a tallow candle, and, we may add, an ordinary jelly, are solids at ordinary temperatures, since their shapes are not permanently altered under the action of gravity.

**Strain and Stress.**—When a change is produced in the relative positions of the constituent particles of a body, the body is said to be **strained**, or to have experienced a **strain**. The external force, the application of which produces the strain, is termed the **stress** on the body. When the strained body is in equilibrium, the stresses must just balance the restoring forces called into play by the strain.

There are two main kinds of strain, each of which is produced by a particular kind of stress.

1. **Compressional Strain.**—When the volume of a body is altered, without the production of any change of shape, the strain is termed **compressional** or **dilatational**, according as the volume is diminished or increased. In this class of strain the angle between any two lines in the body remains unaltered. A cube, of which all three dimensions are increased or decreased equally, may be mentioned as illustrating this kind of strain.

To produce a compressional strain in a cube of a substance which possesses the same elastic properties in all directions, equal stresses must be applied normally to all of its surfaces. The **compressional**

**elasticity** of the substance is defined as ***the ratio of the applied stress, per unit area, to the strain (or alteration in volume) produced in each unit of volume.*** Thus, if a uniform normal force of F dynes per sq. cm., applied to a body of volume $v$, produces a decrease of volume[1] equal to $dv$, then the coefficient of compressional elasticity of the substance, $\epsilon$, is given by the equation—

$$\epsilon = F \div \frac{dv}{v} = \frac{Fv}{dv}.$$

A medium which has the capacity of resisting compression can transmit compressional waves. It can be shown[2] that if $\rho$ is the density of the medium, the velocity of transmission of compressional waves will be equal to $(\epsilon/\rho)^{\frac{1}{2}}$.

2. **Shearing Strain.**—When the shape of a solid is altered in such a manner that each of a series of parallel planes remains undistorted, while the relative positions of the planes are changed, the solid is said to have experienced a **shear**, or a **shearing strain.** If a rectangular parallelopipedon is built up of equal rectangular sheets of thin millboard, then the nature of a shear may be illustrated by displacing these sheets parallel to themselves, as indicated in Fig. 143. If the relative displacement of planes at equal distances from each other is the same throughout the solid, the shear is said to be uniform. The magnitude of a shear is measured by the relative displacement of planes at unit distance apart.

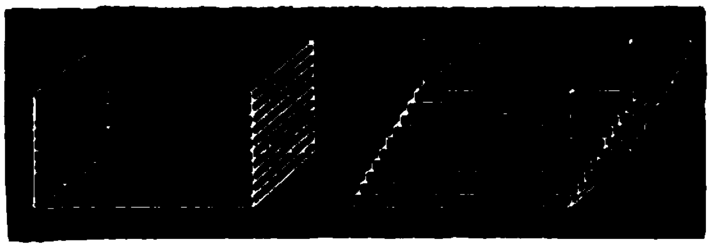

FIG. 143.—Characteristics of a Shearing Strain.

A solid in which a shearing strain produces a restoring force proportional to the shear, is said to possess **rigidity**, or **shear elasticity.** A perfectly rigid body is one in which the production of the smallest conceivable shear would produce an infinite restoring force, and would therefore need the application of an infinite shearing stress. In order to produce a shear between two parallel planes of a substance of finite rigidity, two equal forces must act in opposite directions in these planes. These forces are termed **shearing stresses.**

Let AC (Fig. 144) represent an oblique parallelopipedon, produced by shearing the upper face of the rectangular parallelopipedon AB

[1] The symbol $dv$, taken *as a whole*, is used to indicate a small increase or decrease of volume; it thus represents the *difference produced in v*. The symbol $dv$ does not mean $d \times v$.

[2] See, for instance, *Heat for Advanced Students*, by the author, p. 325.

through a distance BC. Let this require the application to the upper face of a stress F parallel to BC, and an equal and opposite stress to the lower face. Let **a** be the area of either the upper or lower face; then the stress per unit area is equal to F/**a**. Let the perpendicular distance between the upper and lower faces be equal to **b**, while the relative shear BC between these faces is equal to $s$. The shear between planes at unit distance apart is equal to $s$/**b**. Then $\eta$, the **coefficient of rigidity or of shear elasticity** of the substance, is equal to *the ratio of the shearing stress per unit area, to the shear produced between planes at unit distance apart;* therefore—

FIG. 144.—Stresses and Strains in a Sheared Solid.

$$\eta = \frac{\mathrm{F}}{\mathbf{a}} \div \frac{s}{\mathbf{b}} = \frac{\mathrm{F}\mathbf{b}}{s\mathbf{a}}.$$

If the shearing stress is removed, the elastic solid springs back to its original shape. In this process matter is set in motion, and the sheared medium must therefore possess potential energy. The value of this energy may be easily found; it is equal to the work done in producing the shear. Now, the restoring force increases uniformly from zero to F, as the shear increases from zero to $s$; thus, the average restoring force during the production of the shear $s$ is equal to F/2; since this average force is overcome through a distance $s$, the work done is equal to F$s$/2. But $\mathrm{F} = (\eta \mathbf{a} s)/\mathbf{b}$; consequently the potential energy possessed by the sheared parallelopipedon is equal to $\eta \mathbf{a} s^2/2\mathbf{b}$. The volume of the body is equal to **ab**; therefore the potential energy per unit volume is equal to $\frac{1}{2}\eta\left(\frac{s}{\mathbf{b}}\right)^2$. But $s$/**b** is equal to the shear between planes at unit distance apart; if $s/\mathbf{b} = \mathrm{S}$, we have the final result that the **potential energy per unit volume** of the sheared medium is equal to $\eta\mathrm{S}^2/2$.

When a body is compressed or extended in one direction, the dimensions of the body at right angles to this direction remaining unaltered, the volume of each element of the body is altered, while the shape of the element is also modified. Such a strain is said to be ***longitudinal.*** The elasticity of a body for longitudinal strains depends, not only on

the compressional elasticity, $\epsilon$, but also on the rigidity, $\eta$. Its value can be shown to be equal to $\left(\epsilon + \frac{4}{3}\eta\right)$.

**Transverse Waves in an Elastic Solid.**—Imagine an elastic solid, such as a jelly, bounded in one direction by a plane surface, but extending indefinitely in all other directions. Now let the boundary surface be displaced through a small distance parallel to itself. The first effect will be to produce a uniform shear between the boundary plane and any other parallel plane at a small distance within the medium. The resulting reaction will produce a displacement of this second plane, and this displacement will produce a shear between the second and a third plane still further within the medium, resulting in a displacement of this third plane. Following this process mentally, we see that the displacement initially imposed on the boundary plane will be handed on from plane to plane of the elastic solid. In each case the direction of displacement of a plane is parallel to itself; the disturbance is transmitted in a direction at right angles to the direction of displacement.

If we now imagine that the boundary surface is caused to execute a S.H.M., in any direction in its own plane, every other plane parallel to it will execute a S.H.M. in a parallel direction. All particles in one of these planes will, at any instant, be displaced from their positions of equilibrium by the same amount, and will be in the same phase. This form of disturbance transmitted through an elastic medium is said to constitute a **plane wave train**. The plane, perpendicular to the direction of wave transmission, to which the disturbance has just reached, is termed the **wave front**; the same term is sometimes used to denote a plane containing particles in the same phase, whatever position in the wave train it may occupy.

The characteristic features associated with the transmission of transverse waves through an elastic solid may be better appreciated by referring to Fig. 145. This represents a section of an elastic solid by a plane parallel to the direction of wave propagation and to that of the displacements constituting the waves. Particles of the medium which, in their undisturbed state, lay in the broken line (which we shall term the *axis*) extending from left to right, are displaced at a particular instant during the passage of a wave train so as to lie on the thickly-outlined wave curve. The displacement of each particle is perpendicular to the

axis, or to the direction of wave propagation. The fine shading lines represent the displacements of particles which lay, in the undisturbed state of the medium, in lines parallel to the axis. In this class of wave propagation it is necessary to fix our attention on *elements* of the medium, each comprising a considerable number of particles. Fig. 146 represents five equal elements in the path of a wave train. All of these, in the undisturbed state of the medium, had the forms of equal

FIG. 145.—Transverse Wave Displacements in an Elastic Solid.

rectangular parallelopipeda lying along the axis. It is obvious that, during the passage of a wave train, the elements near the crests and troughs of the waves are distorted the least. A *very small* element, lying exactly at the crest or trough of a wave, will be undistorted, and will therefore possess no potential energy; it will also, for the instant, be stationary, and will therefore possess no kinetic energy. An element passing through its position of equilibrium will be submitted to the maximum of distortion, and will therefore possess the maximum amount of potential energy; it will also be moving with the maximum velocity, and will thus possess the maximum amount of kinetic energy.

Consequently, as an element recedes from its position of equilibrium, its kinetic and potential energies decrease together, until the maximum displacement is attained, when the energy of the element has fallen to zero. This energy has not been lost, but has been handed on to elements more in advance. Equal elements separated by a wave-length, or any whole number of wave-lengths, are moving with equal velocities in the same direction, are equally displaced in the same direction, and possess equal amounts of energy. Equal elements separated by half a wave-length, or any odd number of half wave-lengths, are moving with equal velocities in opposite directions, their displacements are equal in magnitude but opposite in sign, and their energies are equal. It should

Fig. 146.—Elements of an Elastic Solid, in the Path of a Transverse Wave Train.

be noticed that the volume of an element remains unaltered (compare Fig. 144), since its faces perpendicular to the direction of wave propagation are unaltered in shape or area, and remain at a uniform perpendicular distance from each other. Each element is subjected to a shear perpendicular to the direction of wave transmission.

**Velocity of Transmission of Transverse Waves in an Elastic Solid.**—During the transmission of waves there is no progressive motion of the medium; its constituent elements merely move to and fro across the direction of transmission, while the waves travel forward. Let the waves be transmitted with a velocity V in the direction of the horizontal arrow (from left to right), Fig. 146.

Let us imagine the medium to be set in bodily motion, with a velocity V, from right to left. The *waves* will now remain

stationary in space; an *element of the medium* which, if there were no waves, would move from right to left along the broken line (Fig. 146), with a velocity V, will now follow the course of the wave curve, and successively acquire the position and forms shown in Fig. 146. In passing the crest or trough of a wave, the element will for an instant be moving with a velocity equal to V; since its path is curved, a centrifugal force is called into play, and this must be just balanced by the shearing stresses exerted on its front and rear surfaces by the elements which respectively precede and follow it. By equating these quantities, we can obtain an expression for V in terms of the properties of the medium.

Let B (Fig. 147) represent the crest of the wave curve ABC, which is traversed by the centre of gravity of the small element of which EDFG is the section. Let AB = BC. Join AC, and draw BM perpendicular to AC. Through the three neighbouring points A, B, and C draw a circle, BLM. The centre of this circle will lie on BM. Then, if the points A, B, C, are very close together, the element EDFG will for an instant be travelling along the arc CBA of the circle BLM. Let the area of the end faces, of which EG and DF are the sections, be equal to a. Also, let the perpendicular distance AC between these faces be equal to $2b_1$. Then the volume of the element EDFG will be equal to $2ab_1$, and if $\rho$ is the density (mass per unit volume) of the medium, the mass of the element will be equal to $2ab_1\rho$. Let $r$ be the radius of the circle BLM. Then the **centrifugal force** acting on the element will be equal to $(2ab_1\rho V^2)/r$ (p. 250).

FIG. 147.—Element at Crest of Wave.

The **shearing stresses** acting on the faces of which EG and DF are sections, will be equal. Let DHKF and EGRS represent the sections of the equal elements which are respectively following and preceding the original element in its path along the wave curve. The points C and N will be the centres of opposite faces of the element DHKF, while A and P will be the centres of the opposite faces of the element EGRS.

Join P and N by a straight line, and let its length be equal to $2\mathbf{b}_2$; let PN cut the diameter BM in a point at a distance $\mathbf{c}_2$ from B. Also, let the straight line AC cut BM in a point at a distance $\mathbf{c}_1$ from B. Then, the shear between the faces DF and HK will be equal to $\mathbf{c}_2 - \mathbf{c}_1$. Further, the distance between the opposite faces DF and HK will be equal to $(\mathbf{b}_2 - \mathbf{b}_1)$, so that the shear per unit distance between the faces DF and HK will be equal to $(\mathbf{c}_2 - \mathbf{c}_1)/(\mathbf{b}_2 - \mathbf{b}_1)$.

Since the lines AC and PN will be at right angles to BM, we have, by a well known property of a circle—

$$\mathbf{b}_2^2 = \mathbf{c}_2(2r - \mathbf{c}_2) = 2\mathbf{c}_2 r - \mathbf{c}_2^2 = 2\mathbf{c}_2 r,$$

since $\mathbf{c}_2$ is very small, and we may neglect $\mathbf{c}_2^2$ in comparison with $(2\mathbf{c}_2 r)$.

Similarly—

$$\mathbf{b}_1^2 = \mathbf{c}_1(2r - \mathbf{c}_1) = 2\mathbf{c}_1 r - \mathbf{c}_1^2 = 2\mathbf{c}_1 r.$$

$$\therefore\ \mathbf{b}_2^2 - \mathbf{b}_1^2 = 2r(\mathbf{c}_2 - \mathbf{c}_1).$$

$$(\mathbf{b}_2 - \mathbf{b}_1)(\mathbf{b}_2 + \mathbf{b}_1) = 2r(\mathbf{c}_2 - \mathbf{c}_1),\ \therefore\ \frac{\mathbf{c}_2 - \mathbf{c}_1}{\mathbf{b}_2 - \mathbf{b}_1} = \frac{1}{2}\frac{(\mathbf{b}_2 + \mathbf{b}_1)}{r}.$$

When the distance between opposite faces of DHKF is very small, we may write $(\mathbf{b}_2 + \mathbf{b}_1) = 2\mathbf{b}_1$, without any sensible error. Then—

$$\frac{\mathbf{c}_2 - \mathbf{c}_1}{\mathbf{b}_2 - \mathbf{b}_1} = \frac{2\mathbf{b}_1}{2r} = \frac{\mathbf{b}_1}{r}.$$

Thus the shear, per unit distance, between the faces DF and HK, is equal to $\mathbf{b}_1/r$. Let F be the shearing stress on either of these faces. Then (p. 268),

$$\frac{\mathrm{F}}{\mathbf{a}} \div \frac{\mathbf{b}_1}{r} = \eta,$$

where $\eta$ is the simple rigidity of the elastic medium

$$\therefore\ \mathrm{F} = \eta\frac{\mathbf{a}\mathbf{b}_1}{r}.$$

The force F will also be equal to the reaction which the sheared element DHKF exerts on the element EDFG. The sheared element EGRS will exert an equal reaction on EDFG, and the sum of these reactions, equal to 2F, must just balance the centrifugal force called into play by the motion of EDFG along the circular arc CBA. Thus—

$$2\mathrm{F} = 2\eta\frac{\mathbf{a}\mathbf{b}_1}{r} = \frac{2\mathbf{a}\mathbf{b}_1\rho\mathrm{V}^2}{r}.$$

$$\therefore\ \mathrm{V}^2 = \frac{\eta}{\rho},\ \text{and}\ \mathrm{V} = \sqrt{\frac{\eta}{\rho}}.$$

Thus, **the velocity of transmission of transverse waves through an elastic solid medium is equal to the square root of the ratio of the shear elasticity, to the density, of the medium.**

In obtaining this result we have made no assumption as to the wave-length or period of the transmitted waves. Thus, **the velocity is the same for waves of all lengths.** A particle of an elastic solid has no free period of vibration, and the restoring force called into play by its displacement is not proportional to the displacement, but to the difference between that and the displacements of neighbouring particles. If the boundary plane of an elastic solid is constrained to execute a S.H.M., each particle of the elastic solid will execute a S.H.M.; but this is merely an instance of forced vibrations, since a motion of any kind whatever would be transferred from the boundary plane to the constituent particles of the elastic medium. As we have seen, however, a disturbance of any kind whatever can be resolved into a series of simple harmonic wave trains, so that results obtained for the transmission of the latter can be applied to the general case of the transmission of an arbitrary disturbance.

**Average Energy per Unit Volume of an Elastic Solid transmitting Simple Harmonic Waves.**—As already pointed out, an element of an elastic solid possesses the maximum amount of energy, both potential and kinetic, as it passes through its position of equilibrium (p. 270). Further, if the waves are of the simple harmonic type, each element executes a S.H.M. about its position of equilibrium, so that its velocity on passing through that position, is equal to $2\pi a/T$, where $a$ and T respectively stand for the amplitude and period of the S.H.M. Let the volume of an element be equal to $v$, while $\rho$, as before, is equal to the density of the medium. Then the mass of the element is equal to $\rho v$, and its kinetic energy on passing through its position of equilibrium, is equal to $\frac{1}{2}.\rho v.(2\pi a)^2/T^2$ (p. 241); thus **the maximum kinetic energy per unit volume** of the element is equal to—

$$\tfrac{1}{2}\rho\frac{(2\pi a)^2}{T^2}.$$

As an element recedes from its position of equilibrium, its kinetic energy diminishes to zero as the point of its maximum displacement is approached (p. 271). In a wave train compris-

ing a number of waves, there will, at any instant, be elements displaced to various extents, and therefore possessing various fractions of the maximum kinetic energy. Consequently, the **average kinetic energy per unit volume of the elastic medium** in the path of the wave train, will be equal to half the above maximum value, or to—

$$\tfrac{1}{4}\rho\,\frac{(2\pi a)^2}{T^2}.$$

An element on passing through its position of equilibrium will be distorted (Fig. 146), and will thus possess potential energy. A value has already been found (p. 268) for the potential energy per unit volume of a sheared element of an elastic medium; this is equal to $\left\{\frac{\eta}{2} \times (\text{shear per unit distance})^2\right\}$.

Let ABCD (Fig. 148) represent the section of an element passing through its position of equilibrium, EF being a short length of the wave curve. From E draw EG perpendicular to AD. Then, it is obvious that the shear between the faces AD and BC is equal to FG, and the **shear per unit distance** between AD and BC is equal to FG/GE. Let PQ represent the axis of wave transmission, and let $\theta$ be the angle at which the wave curve cuts the axis PQ. Then, we have—

$$FG/GE = \tan\theta.$$

FIG. 148.—Sheared Element.

But, as proved on p. 261, $\tan\theta = 2\pi a/VT$, where $a$, V, and T respectively stand for the amplitude, the velocity of wave transmission, and the period of the waves. Consequently, the **potential energy per unit volume** of the element ABCD is equal to—

$$\frac{\eta}{2}\left(\frac{FG}{GE}\right)^2 = \frac{\eta}{2}\tan^2\theta = \frac{\eta}{2}\,\frac{(2\pi a)^2}{V^2T^2}.$$

From reasoning similar to that employed with respect to the kinetic energy of the medium, it follows that **the average potential energy per unit volume of the elastic medium** in the path of the wave train is equal to—

$$\tfrac{1}{4}\eta\cdot\frac{(2\pi a)^2}{V^2T^2}.$$

But $\eta = \rho V^2$ (p. 273). Consequently, **the average potential energy per unit volume is equal to the average kinetic energy per unit volume, and the average energy of both kinds per unit volume is numerically equal to—**

$$\frac{1}{2}\rho\frac{(2\pi a)^2}{T^2}.$$

Some important consequences may be deduced from this result.

1. With waves of a given period, T, the **energy per unit volume** of the elastic medium is **proportional** to $\rho a^2$, that is, to the product of the density and the square of the amplitude.

2. The **energy transmitted per second** normally through an imaginary surface of 1 sq. cm. area, will be that corresponding to V c.c. of the medium, or $V\rho(2\pi a)^2/2T^2$.

**Waves of Circular Displacement.**—Let us suppose that the plane boundary surface of an elastic solid is constrained to move in such a manner that each point in it describes a small circle in the plane of the surface. This motion is equivalent to two S.H.M.'s, of equal amplitudes and periods, but differing in phase by $\pi/2$, executed in directions at right angles to each other (p. 242). These S.H.M.'s will generate corresponding wave trains travelling with equal velocities through the elastic medium, so that each particle of the latter will simultaneously execute two equal S.H.M.'s perpendicular to each other, and differing in phase by $\pi/2$; in other words, each particle of the medium will traverse a small circular orbit, in a plane perpendicular to the direction of wave transmission.

**Spherical Waves.**—When all points in a plane are constrained to execute similar motions, plane waves will be produced in the elastic medium. If, however, a single point in the medium is constrained to execute a S.H.M. in a given direction, then waves will spread out in all directions from that point, so that the wave front assumes a spherical form. If the medium is capable only of transmitting transverse waves, the displacement at any point of the spherical wave must be perpendicular to the radius, or tangential to the surface, of the sphere. But at a point on the spherical wave surface in a line with the direction of displacement at the centre, a displacement perpendicular to the radius would be perpendicular to the displacement at the centre. Since the medium

has perfect freedom of motion, such a displacement cannot be produced, so that **at points in a straight line with the direction of the central displacement the amplitude of the wave will be equal to zero.** The amplitude will have a maximum value at points on the spherical wave surface which are at right angles to the direction of the central displacement.

For a given direction of transmission **the amplitude will be inversely proportional to the distance from the point at which the disturbance originates.** To prove this statement, let us mark off a small element of area $\mathbf{a}_1$ on a spherical surface of radius $r_1$, and draw straight lines from the centre through the boundary of this element; we shall thus obtain a cone which will cut off an area $\mathbf{a}_2$ from a sphere of radius $r_2$. Now $\mathbf{a}_1/\mathbf{a}_2 = r_1^2/r_2^2$. Let $a_1$ and $a_2$ be the respective amplitudes at distances $r_1$ and $r_2$ from the centre. Then the energy traversing the area $\mathbf{a}_1$, at a distance $r_1$ from the centre, in one second, will also traverse the area $\mathbf{a}_2$, at a distance $r_2$ from the centre, in one second. Therefore (p. 276),

$$\frac{\mathbf{a}_1 V\rho\,(2\pi a_1)^2}{2T^2} = \frac{\mathbf{a}_2 V\rho\,(2\pi a_2)^2}{2T^2}; \quad \therefore \quad \frac{a_1^2}{a_2^2} = \frac{\mathbf{a}_2}{\mathbf{a}_1} = \frac{r_2^2}{r_1^2};$$

and

$$\frac{a_1}{a_2} = \frac{r_2}{r_1}.$$

**Elastic Solid with Particles capable of Free Vibrations embedded in it.**—We have already (p. 254) investigated the motion of a particle capable of free vibrations about a position of equilibrium which itself executes a S.H.M. Let us now extend this investigation to the motion of an elastic solid, embedded in which are great numbers of particles capable of free vibrations. To fix our ideas, let us suppose that heavy particles are supported, by means of zigzag springs, inside massless spherical shells (Fig. 149), which, on their external surfaces, are rigidly fixed to the elastic solid. When the elastic solid is at rest, the heavy particles will occupy their positions of *absolute equilibrium* (p. 256), at the centres of the stationary spherical shells. When the elastic solid is displaced, the spherical shells will be carried with it. The

Fig. 149. — Vibrating Particle embedded in Elastic Solid.

first effect produced will be to compress some of the zigzag springs, and to extend others, so that forces will be exerted on the heavy particles proportional to their relative displacements from the centres of the shells. Equal and opposite reactions will be exerted on the shells, and thus on the elastic solid; these reactions will profoundly modify the nature of the waves transmitted through the elastic solid.

The following points of importance may be deduced from Fig. 138 (p. 254):—

1. On starting the waves through the elastic solid, the motions of the embedded particles will at first be of an apparently irregular nature, due to the circumstance that they, at first, simultaneously execute free and forced vibrations. After a time, however, the free vibrations die down (p. 254), and the particles execute only forced vibrations in the period of the waves. The following arguments apply only to the time subsequent to the acquisition of this permanent state.

2. As an element of the elastic solid passes through its position of equilibrium, the heavy particles embedded in it will swing through their positions of absolute equilibrium. At this point no reaction is exerted on the elastic solid by the embedded particles.

3. As an element of the elastic solid recedes from its position of equilibrium, the relative displacements of the particles embedded in it increase, and reach a maximum value when the element reaches its position of maximum displacement at the crest of a wave. At this point the reaction exerted on the elastic solid by the embedded particles reaches a maximum value.

**Velocity of Wave Transmission.**—This can be found in a manner similar to that employed on p. 271, if we make allowance for the reactions of the embedded particles.

Let **F** be the reaction exerted by an embedded particle on an element of the elastic solid when the displacement of the latter is equal to $a$, the amplitude of a wave. If there are $n$ particles distributed uniformly through each unit volume of the medium, the number comprised in the element of which EDFG (Fig. 147), is a section, will be equal to $n \times 2\mathbf{a}\mathbf{b}_1$, where $\mathbf{a}$ and $\mathbf{b}_1$ have the same significations as on p. 272. Let us assume that the embedded particles are so numerous that a considerable number are comprised in the element $EDFG_1$ while their volumes are so small that no appreciable amount of the elastic solid is displaced, in order to make room for them, and no change is produced in the rigidity, $\eta$. As the element EDFG passes the crest of a wave, the centrifugal force, together with the sum of the reactions of the

embedded particles tending to increase the displacement of the element, must be balanced by the shearing stresses on the faces DF and EG. If V denotes the velocity of wave transmission through the elastic solid with the heavy particles embedded in it, we have—

$$\frac{2ab_1\rho V^2}{r} + 2ab_1nF = \frac{2\eta ab_1}{r};$$

$$\therefore \frac{\eta}{\rho} - V^2 = \frac{rnF}{\rho}$$

From p. 256, we find that $F = fa\dfrac{T_1^2}{T^2 - T_1^2}$,

where $f$ is the restoring force per unit displacement of the heavy particle from its position of relative equilibrium, T is the period of the waves, and $T_1$ is the free period of the heavy particle. Consequently, **the value of V will be different for waves of different periods.** Let $\eta/\rho = V_0^2$, where $V_0$ is the velocity of wave transmission through the elastic solid when the heavy particles are not present. Then—

$$V_0^2 - V^2 = ra\frac{nf}{\rho}\frac{T_1^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We must now find the value of $r$, the radius of the circle passing through three nearly coincident points in the neighbourhood of a crest

FIG. 150.—To determine the Radius of Curvature at the Crest of a Wave Curve.

of the wave curve. Let DBE (Fig. 150) represent part of the wave curve obtained in the manner explained in connection with Fig. 140 (p. 260). Let GHKM be the circular path of the tracing point, the motion of which determines the S.H.M. executed by the pencil point. A and C represent two points equidistant from, and on opposite sides of, the

crest B of the wave; then, while the pencil point described the arc ABC of the wave curve on the moving paper riband, the tracing point moved from G through H to K. Join CA, and draw BN perpendicular to CA, cutting the latter in F. Through C, B, and A describe the circle CBAN. The centre of this circle will be on the line BN: let its radius be equal to $r$. Let the time required for the pencil point to describe the arc AB be equal to $t$. Then in this time the paper riband will have moved onward through a distance $Vt$, which is thus the length of the line FA. Let BF $\doteq \delta$. Then, by a property of the circle which we have frequently had occasion to use, we have, if $\delta$ is very small—

$$2r \cdot \delta = (\mathrm{FA})^2 = (Vt)^2. \quad \therefore \delta \doteq \frac{(Vt)^2}{2r} \quad . \quad . \quad . \quad (2)$$

It is obvious that LH will also be equal to $\delta$. Further, the tracing point describes the circle GHKM, of radius equal to $a$, in a time T; thus, it moves with a velocity equal to $2\pi a/T$, and the arc GH, described in the time $t$, will be equal to $2\pi at/T$. If G is very near to H, the straight line GL will be approximately equal to GH. Then, since $(\mathrm{GL})^2 = (\mathrm{HL})(\mathrm{LM}) = (\mathrm{HL})(\mathrm{HM})$ to a first approximation, we have—

$$\left(\frac{2\pi at}{T}\right)^2 = 2a\delta = 2a \cdot \frac{(Vt)^2}{2r}, \text{ from (2)};$$

$$\therefore ra\left(\frac{2\pi}{T}\right)^2 = V^2, \text{ and } ra = \left(\frac{VT}{2\pi}\right)^2.$$

Substituting this value of $ra$ in (1), we obtain—

$$V_0^2 - V^2 = nf\frac{T_1^2}{(2\pi)^2\rho} \cdot V^2\frac{T^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad (3)$$

Let $\dfrac{fT_1^2}{(2\pi)^2\rho} = K$. Then, substituting in (3) and dividing both sides by $V^2$, we obtain the equation—

$$\frac{V_0^2}{V^2} - 1 = n \cdot K\frac{T^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation gives us V (the velocity of wave transmission through the elastic solid containing particles capable of free vibration) in terms of $V_0$ (the velocity of wave transmission in the elastic solid when the heavy particles are absent), T (the period of the waves), and $T_1$ (the time of free vibration of the embedded particles).

We shall discuss this equation in connection with the dispersion of light by material substances. It need only be remarked here that when the right hand side of (4) is positive (as will be

the case when $T > T_1$), V will be less than $V_0$. When, however, $T < T_1$, the right hand side of (4) will be negative, and V will be greater than $V_0$.

If the elastic solid contains two sets of heavy particles, possessing free vibration periods equal to $T_1$ and $T_2$ respectively, the velocity V for transverse waves of period T will obviously be given by the equation,

$$\frac{V_0^2}{V^2} - 1 = n_1 K_1 \frac{T^2}{T^2 - T_1^2} + n_2 K_2 \frac{T^2}{T^2 - T_2^2},$$

where $n_1$ and $n_2$ are the numbers, per unit volume, of particles of the two kinds.

Lastly, let us suppose that the vibrating particles are all similar, but that the motions of each are controlled by three sets of springs of different elasticities. We may suppose, for instance, that the horizontal and vertical springs shown in Fig. 149 have different elasticities, while a third pair of springs, at right angles to the plane of the paper, has yet another elasticity. The vibrating particle will then have three distinct periods of vibration, according as the direction of its vibration coincides with that of the first, second, or third pair of springs. Then, if these particles are arranged regularly in the elastic solid, with the similar springs in the same directions, transverse waves will be transmitted with different velocities, according as the direction of displacement of the waves coincides with that of the first, second, or third pair of springs. In other words, we should have an elastic solid which transmits transverse waves of any given period with a velocity depending on the direction of transmission.

**Average Energy per Unit Volume.**—From the nature of the argument used on p. 254, it is clear that the reactions of the embedded particles produce an apparent increase (when $T > T_1$) in the inertia of the elastic medium.

The embedded particles will themselves possess an amount of energy, which in certain cases may be considerable. But after the acquisition of the permanent state (p. 278), a part only of the energy possessed by a vibrating particle will be interchanged with the elastic medium. This part, which may be termed the **available energy** of a particle, is equivalent to the work done on the medium by a particle as it swings from its position of absolute equilibrium to that of its maximum dis-

placement, while its position of relative equilibrium is displaced through a distance $a$, equal to the amplitude of the waves. The value of this work is given by W, equation (3), p. 256. At a given instant some of the particles in the path of a wave train will be at their positions of maximum displacement, and so will have parted with all of their available energy. Some will be passing through their positions of absolute equilibrium, and will possess the maximum available energy. Between these two extremes there will be particles possessing various fractions of their maximum available energy, so that the **average available energy** possessed by the particles embedded in unit volume of the medium will be equal to $n\mathrm{W}/2$, or

$$nf\frac{a^2}{4}\cdot\frac{\mathrm{T}_1^2}{\mathrm{T}^2-\mathrm{T}_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $n$ is the number of particles per unit volume. This will represent the increase, due to the presence of the vibrating particles, in the kinetic energy per unit volume of the medium.

Proceeding as on p. 275, we find that the **average potential energy per unit volume** is equal to—

$$\frac{\eta}{4}\frac{(2\pi a)^2}{\mathrm{V}^2\mathrm{T}^2}.$$

If the vibrating particles were absent, the average potential energy per unit volume would be equal to the same quantity with $\mathrm{V}_0$, substituted for V. Hence the increase in the average potential energy per unit volume, due to the presence of the vibrating particles, is equal to—

$$\frac{\eta}{4}\frac{(2\pi a)^2}{\mathrm{T}^2}\left(\frac{1}{\mathrm{V}^2}-\frac{1}{\mathrm{V}_0^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But $\eta = \rho\mathrm{V}_0^2$ (p. 272). Substituting in (6) we obtain, by the aid of (4), p. 280,

$$\frac{1}{4}\frac{(2\pi a)^2\rho}{\mathrm{T}^2}\left(\frac{\mathrm{V}_0^2}{\mathrm{V}^2}-1\right) = \frac{1}{4}\frac{(2\pi a)^2}{\mathrm{T}^2}\cdot\rho n\cdot\frac{f\mathrm{T}_1^2}{(2\pi)^2\rho}\frac{\mathrm{T}^2}{\mathrm{T}^2-\mathrm{T}_1^2}$$

$$= nf\frac{a^2}{4}\cdot\frac{\mathrm{T}_1^2}{\mathrm{T}^2-\mathrm{T}_1^2}.$$

This value is equal to that given by (5) for the increase in the average kinetic energy per unit volume due to the presence of the vibrating particles. Hence, **the presence of the vibrating particles increases the kinetic and potential energies of the medium to equal extents.** Thus an apparent increase in the density of the medium is produced; if

$\rho'$ is the effective density of the medium for waves of period T, we have, following the process used on p. 274, and using (5),

$$\tfrac{1}{2}\rho'\frac{(2\pi a)^2}{T^2} = \tfrac{1}{2}\rho\,\frac{(2\pi a)^2}{T^2} + \tfrac{1}{2}nfa^2\frac{T_1^{\,2}}{T^2 - T_1^{\,2}};$$

$$\therefore\quad \frac{\rho'}{\rho} = 1 + \frac{nfT_1^{\,2}}{(2\pi)^2\rho}\,\frac{T^2}{T^2 - T_1^{\,2}} = 1 + nK\,\frac{T^2}{T^2 - T_1^{\,2}}.$$

When the period of the transmitted waves is equal to the free period of the vibrating particles (*i.e.* $T = T_1$) this value of $\rho'$ becomes infinite. For other values of T, $\rho'$ will have definite values, so that **the effect of the vibrating particles is to endow the medium with an effective density $\rho'$, which varies with the period of the transmitted waves.**

**Phase Change on Reflection.**—Let us suppose that plane waves of transverse displacement, after traversing an elastic medium, are incident normally on the plane boundary surface of the latter. First, let us suppose that this boundary surface is immovable, as if it were fixed to a rigid wall. In this case the displacement at the boundary surface must be equal to zero. With a single train of waves this condition could not be satisfied. On the other hand, when two similar wave trains travel with equal velocities in opposite directions, stationary points, or nodes, are formed in the resultant stationary wave train (p. 264). If one of these nodes is formed at the boundary surface, there will be no displacement there. Consequently the incident wave train must, at the boundary surface, originate a reflected wave train of equal amplitude, the two producing a node at the boundary surface. But, at a node, the displacements due to the component waves must always be equal in magnitude and opposite in directions, or must differ in phase by $\pi$. Consequently, **the reflected and incident wave trains must differ in phase by $\pi$.**

If a wave train transmitted through an elastic medium is incident on a surface separating the medium from another of much greater density, the effect will be practically the same; the amplitude of the waves transmitted into the second medium will be excessively small, and the resultant displacement at the separating surface must be excessively small, so that a reflected wave train differing in phase by $\pi$ from the incident train must be produced. More generally, **when waves are incident on the**

**boundary surface of a second medium which is denser than the first, the reflected wave train will differ in phase by $\pi$ from the incident train.** On comparing Figs. 142 and 146, it will readily be seen that a node is a point at which the distortion of the medium attains a maximum value. This distortion can only occur when the second medium is denser, or set in motion less readily, than the first medium. A difference of *effective density* (p. 283) will serve the same purpose as a difference of real density.

When waves are incident on a *free* boundary surface, a node cannot be formed there, since the constraint necessary to produce distortion will be absent. In this case the incident waves (which convey energy up to the surface) and the reflected waves (which carry it away) must combine to form an *antinode* at the boundary surface. Since at the surface the displacements of the two wave trains are equal and in the same direction (p. 265), the phases of the incident and reflected wave trains must be equal. More generally, **when waves are incident on the boundary surface of a second medium, which is less dense than the first, the phases of the incident and reflected wave trains will be equal.**

The nature of the phase change produced by reflection is often explained by analogy with the impact of elastic spheres. Let two elastic spheres be suspended by fine filaments so as to form pendulums, and let the spheres just touch when in their positions of equilibrium. If one of the spheres is displaced, and then allowed to impinge on the other, both spheres will, after impact, move on together, if the impinging sphere is heavier than the sphere which is struck. In this case the phase of the S.H.M. executed by the impinging sphere is not altered by impact. If, on the other hand, the impinging sphere is lighter than the sphere struck, the two will move in opposite directions after impact, the lighter sphere rebounding, so that its direction of motion is reversed by the impact. This corresponds to a change of phase amounting to $\pi$ in the S.H.M. executed by the impinging sphere.

In a stationary wave train formed in an elastic solid, the energy will be wholly kinetic at the instant when the displacement curve takes the form of a straight line (Fig. 142), and wholly potential when the antinodes have reached their position of maximum displacement. An element of the medium situated at an antinode will never possess any potential energy; it will possess the maximum amount of kinetic energy as it swings

through its position of equilibrium, while it will possess no energy at all at its position of maximum displacement. An element at a node will possess the maximum amount of potential energy at the instant when the antinodes have acquired their maximum displacements, and will possess no energy at all when the antinodes swing through their positions of equilibrium.

## QUESTIONS ON CHAPTER XII

1. Define a simple vibration. Show by a diagram how to exhibit the result of adding together two simple vibrations of different frequencies.

2. Assuming that the velocity of the bob of a simple pendulum at the mid point of its swing is equal to the velocity of a point moving uniformly round a circle of radius equal to the amplitude of the pendulum's vibration, and with an equal period, and assuming also that the kinetic energy of the bob in the middle of its swing is numerically the same as its potential energy at the end of its swing, prove that the number of swings it makes in one second is $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$; where $g$ is the acceleration of gravity and $l$ is the length of the pendulum.

3. Show how to find the resultant motion obtained by compounding together two equal uniform circular motions, of the same period, in the same plane: (*a*) when the two motions are in the same sense; (*b*) when they are in opposite senses. What will be the resultant motion if the two circular motions are in opposite senses, and differ very slightly from one another in period?

# CHAPTER XIII

## THE WAVE THEORY OF LIGHT

**The Luminiferous Ether.**—The fact that light is transmitted more slowly in a highly refracting medium (such as water) than in air or in a vacuum, gives us decisive evidence against the corpuscular theory of light. Our only alternative is to seek an explanation of the phenomena of light in terms of waves. But waves can only be propagated through a continuous medium. Consequently, in order to account for the arrival at the earth of light from the stars, we must assume the presence, in the intervening space, of a continuous medium capable of transmitting waves. This medium is termed the **Luminiferous Ether**. Its properties cannot be directly apprehended by the aid of our senses, but must be inferred from the properties of the waves transmitted through it. In the first place, it must possess inertia, or the property of acquiring kinetic energy when set in motion. In other words, **the ether must possess a definite density**. It must also possess the property of acquiring potential energy when strained. In other words, **the ether must possess elasticity**. In deciding as to the nature of this elasticity, we must be guided by the experimental evidence which we possess, relating to the behaviour of light in various circumstances.

**Wave Propagation.**—As far as we know, light can only be generated by the agency of material bodies. Thus, the sun, and (presumably) the stars, are material bodies emitting light. A candle or gas flame, a red-hot poker, the incandescent filament of an electric glow lamp, and a glow-worm, are all instances of material bodies which, under appropriate conditions, generate light, or produce waves in the luminiferous ether.

Let us suppose that light is generated by material agency at a point O (Fig. 151). Let us also assume that the properties of the ether are similar in all directions. Then, the ethereal waves produced will spread out from O as centre, in the form of spherical sheets (p. 276). Let *ab* represent part of a particular wave front (p. 269); this will be an imaginary surface, described through contiguous particles of the ether which are in the same phase of vibration. To fix our ideas, we may suppose that all particles in this surface are just moving through their positions of equilibrium in directions which, with regard to the wave front, are similar. After a time, the particles in the imaginary surface, of which *a′b′* is a section, will be moving in a similar manner. We may say that in this time the wave front has travelled from the position *ab* to that of *a′b′*. Since the disturbance originated at O, we may, for certain purposes, look on the wave front as a condition of the ether which is transferred from O, to one after another of a series of imaginary spherical surfaces with O as centre, the radius of the spherical surface reached at any instant being equal to the distance through which light can travel in the time which has elapsed since the wave front originated at O. But, from another point of view, we must follow Huyghens in considering each point in a wave front as an independent source of disturbance which originates a secondary wave, or wavelet, the new wave front being produced by the combined effect of the innumerable wavelets which originated at points in the old wave front. With M, M′, . . . . as centres, describe small spheres to represent the wavelets generated at points in the spherical wave front *ab*. These wavelets touch a sphere *a′b′*, of which the centre is at O; this spherical surface will include all the particles to which the displacement and motion characterising the old wave front has been communicated, and will thus constitute the new wave front.

FIG. 151.—Propagation of Waves.

If the distance of the wave front from the point of origin, O, is very great, a finite portion of the wave front will be sensibly plane,

just as the surface of a small lake, which is really a portion of a spherical surface concentric with the earth, is sensibly flat. Thus, the waves reaching us from the stars are sensibly plane.

The process described above represents the method of propagation of waves in general. A difficulty now presents itself, which is of a character so serious, that it led Newton to discard Huyghens's wave theory of light. It is well known that sound waves can bend round corners, while the propagation of light is sensibly rectilinear. Before proceeding to the explanation of the particular phenomena of light, this difficulty must be disposed of.

**Huyghens's Zones.**—Let plane waves be propagated in the direction from P to O (Fig. 152). The various wave fronts will be perpendicular to the line OP, and will successively pass through the imaginary plane ABCD, perpendicular to OP. Thus, at any instant the displacements of the various particles in the plane ABCD will be exactly similar and equal, or the particles will be in the same phase of vibration. If the waves are of the simple harmonic type (p. 269), each particle in the plane ABCD will execute a S.H.M. as a plane wave front moves, from the position occupied by ABCD, through a distance equal to one wave-length. But the vibration of each particle will generate a spherical wavelet, which spreads out from the particle as centre, and successively passes through all points in the medium. At any given instant wavelets originating from the various particles in the plane ABCD will be passing through the point O, and the resultant disturbance at O will be due to the combined action of all of these wavelets. It is obvious that the phases of the wavelets will differ, for all travel with the same velocity, and some originated at points in

FIG. 152.—Huyghens's Zones.

the plane ABCD which are more distant from O than others. Let OP, the distance of the nearest point, P, of the plane ABCD, be equal to $b$, and let $\lambda$ be the length of the transmitted waves: With O as centre, and radius equal to $b + \frac{\lambda}{2}$, describe a sphere cutting the plane ABCD in the circle $M_1$. Let wavelets from P and from points on $M_1$ simultaneously pass through O. Then, since points on the circle $M_1$ are at a distance from O which is greater by $\lambda/2$ than the distance $b$ of the point P, the wavelets originating from points on $M_1$ must have started T/2 seconds earlier than the wavelet from P, where T is the period of the waves. Consequently the wavelets from $M_1$ will differ in phase by $\pi$ from the wavelet from P, and wavelets originating from any point within the circle $M_1$ will differ in phase by less than $\pi$ from the wavelet from P. The space enclosed by the circle $M_1$ is termed the **first half-period zone.**

With O as centre, and radius $(b + \lambda)$, describe a second sphere cutting ABCD in the circle $M_2$. Then wavelets from points on $M_2$ will differ in phase by $\pi$ from the wavelets from $M_1$, and by $2\pi$ from the wavelet from P. The space between the circles $M_1$ and $M_2$ is termed the **second half-period zone.**

In a similar manner, with radii respectively equal to $\left(b + \frac{3\lambda}{2}\right)$, $(b + 2\lambda)$, $\left(b + \frac{5\lambda}{2}\right)$, . . . . etc., describe spheres dividing the plane ABCD into an indefinite number of half-period zones.

We must now determine the area of a half-period zone. Let AB (Fig. 153) represent a section of the plane ABCD (Fig. 152) by a plane passing through the points O and P. Then the area of the first half-period zone is equal to $\pi \times (PM_1)^2$.

Also—

$$(PM_1)^2 = (OM_1)^2 - (OP)^2 = \left(b + \frac{\lambda}{2}\right)^2 - b^2 = b^2 + b\lambda + \left(\frac{\lambda}{2}\right)^2 - b^2$$
$$= b\lambda + \left(\frac{\lambda}{2}\right)^2.$$

Now when $\lambda$, the wave-length of the undulations, is very small, $\lambda^2$ will be negligible in comparison with $b\lambda$. In that case—

$$(PM_1)^2 = b\lambda,$$

**and the area of the first half-period zone is equal to $\pi b\lambda$.**

The combined areas of the first and second half-period zones will be equal to $\pi \times (PM_2)^2$. As before—

$$(PM_2)^2 = (OM_2)^2 - (OP)^2 = (b + \lambda)^2 - b^2 = b^2 + 2b\lambda + \lambda^2 - b^2 = 2b\lambda,$$

neglecting $\lambda^2$ in comparison with $2b\lambda$. Thus, the combined areas of the first and second half-period zones are equal to $2\pi b\lambda$, or to twice the area of the first zone, and **the area of the second half-period zone is equal to that of the first, or to $\pi b\lambda$.**

Proceeding in a similar manner, we find that the combined areas of the first three half-period zones are equal to $\pi \times (PM_3)^2 = 3\pi b\lambda$, and since the combined areas of the first and second zones are equal to $2\pi b\lambda$, it follows that **the area of the third half period zone is equal to $\pi b\lambda$, or to that of the first zone.**

FIG. 153.—Half-period Zones.

Similarly, it is easily proved that when $\lambda$ is small, the areas of all of the half-period zones are equal.

It may be noticed that the radii of the circles bounding the various zones are respectively equal to $\sqrt{b\lambda}$, $\sqrt{2b\lambda}$, $\sqrt{3b\lambda}$, $\sqrt{4b\lambda}$, . . . . and are thus proportional to the square roots of the natural numbers.

The displacement D at the point O will be the resultant of the displacements due to the wavelets arriving there from various points in the plane ABCD. Let $d_1$ be the displacement at O due to the combined action of the wavelets from points in the first half-period zone. The phases of these wavelets will vary between 0 and $\pi$; if we knew the amplitude and phase of each wavelet, we could find their resultant by the method explained on p. 248. As it is, there is no great difficulty in seeing that the phase of the resultant will approximately be equal to the mean of the phases of the components, and will thus differ by $\frac{\pi}{2}$ from the phase of the wavelet from P.

The displacement at O due to the combined action of wavelets

from the second half-period zone may be written equal to $-d_2$. For the phases of these wavelets will vary between $\pi$ and $2\pi$, and the phase of the resultant will be approximately equal to $\frac{3\pi}{2}$, so that it will differ by $\pi$ from the phase of $d_1$; hence the minus sign, showing that the displacement is opposite to that of $d_1$.

The displacement at O due to the combined action of the wavelets from the third half-period zone may be written down as $d_3$. The phases of these wavelets vary between $2\pi$ and $3\pi$, so that the phase of the resultant is equal to $5\pi/2$, which differs by $2\pi$ from the phase of $d_1$. A positive sign is prefixed, since harmonic displacements differing in phase by $2\pi$ are in the same direction. The displacements due to wavelets from the fourth, fifth, sixth, &c., zones, may, for similar reasons, be written down as respectively equal to $-d_4$, $+d_5$, $-d_6$, .... &c. Hence—

$$D = d_1 - d_2 + d_3 - d_4 + d_5 - d_6 + d_7 - \ldots \quad (1)$$

It will now be shown that **the displacement due to wavelets from any half-period zone is numerically equal to half the sum of the displacements due to the wavelets from the preceding and succeeding zones.**

Since the half-period zones are all equal in area, an equal quantity of energy will be transmitted through each. Thus, the amplitudes of spherical wavelets *of equal radii* will be equal (p. 276), while in general the amplitudes of different wavelets will be inversely proportional to their radii. Since the distance of a zone from the point O increases with the order of the zone, the numerical values of $d_1$, $d_2$, $d_3$, $d_4$ . . . , though nearly equal, will be in a descending order of magnitude. Thus, $d_2$ will be slightly smaller than $d_1$, and slightly greater than $d_3$, and, to a close approximation, $d_2 = (d_1 + d_3)/2$.

Similarly, $d_4 = (d_3 + d_5)/2$, and $d_6 = (d_5 + d_7)/2$, &c.

Now, (1) may be written—

$$D = \frac{d_1}{2} + \left(\frac{d_1 + d_3}{2} - d_2\right) + \left(\frac{d_3 + d_5}{2} - d_4\right) + \left(\frac{d_5 + d_7}{2} - d_6\right) + \ldots\ldots$$

Each of the terms enclosed in brackets is equal to zero, so that **the resultant displacement at O, due to wavelets from all points of the plane ABCD (Fig. 152), will be equal to half the displacement due to the wavelets from the first half-period zone.**

It may be noticed, in passing, that the phase of the resultant displacement at O will be equal to that of $d_1$, and this, as proved above, is behind that of the wavelet from the point P by $\frac{\pi}{2}$.

**Rectilinear Propagation of Light.**—Experience shows that a very small body, even when placed at some distance in front of the eye, will entirely hide a star from view. Consequently light does not *appreciably* bend round corners, as sound does. The above investigation shows that this result is quite consistent with the wave theory of light, **provided that the length of a light wave is very small in comparison with ordinary magnitudes.** We have found that the phases of the displacements due to wavelets from successive half-period zones differ by $\pi$, and are therefore in opposite directions. The magnitudes of the displacements due to wavelets from successive zones decrease with the order of the zone, the order of a zone being its number, counting that of the central zone as unity. But as the order of the zones increases, their effects at the point O (Fig. 152) become more and more nearly equal in magnitude, so that the displacement due to wavelets from any zone of high order is just cancelled by that of the next succeeding zone. Thus, if an obstacle is sufficiently large to cover up the first few half-period zones, the wavelets from the remaining zones will just cancel each other, and no light will be perceived by the eye.

In order to appreciate the difference between the propagation of sound and light waves, it may be remarked that the wave-length, $\lambda$, corresponding to the middle C of a pianoforte, is about 120 cm. If we suppose that the plane ABCD is situated at a distance $b = 100$ cm. from the ear, the radius of the tenth half-period zone, equal to $\sqrt{10b\lambda}$, will be approximately equal to 300 cm. It has been found that the wave-length corresponding to the yellow light emitted by a Bunsen flame, into which some common salt has been introduced, is roughly equal to $6 \times 10^{-5}$ cm. If we suppose the plane ABCD to be situated at 100 cm. from the eye, the radius of the tenth half-period zone will be equal to $\sqrt{\cdot 06} = 0\cdot 1$ cm. Thus, the screening action, with regard to light, of an obstacle only a few millimetres in diameter, is equivalent to the screening action, with regard to sound, of an obstacle several metres in diameter.

The screening action of a comparatively small obstacle with regard to sounds of high pitch can easily be observed.

**Expt. 54.**—Hold a watch at a distance of 30 or 40 cms. from the ear, and notice the effect of placing your flat hand, or a small card, at a distance of from 5 to 10 cm. in front of your ear. The sound is appreciably deadened, the waves of short length being most affected, so that much of the metallic ring is destroyed. The shrill whistling sound produced when a locomotive engine blows off steam may be deadened in a similar manner.

**Light Rays.**—Let AB (Fig. 154) represent one of a series of plane light waves travelling parallel to PO. The resultant displacement at any point, O, is equal to half the displacement due to wavelets from the first half-period zone surrounding P. The diameter of this zone will be very small, since the wave-length of the light is small, so that the disturbance reaching O has travelled sensibly along the line PO. Thus, PO is the **light ray** reaching O. It consequently appears that a train of plane light waves is equivalent to a pencil of parallel rays. Similarly it can be proved that a train of spherical waves is equivalent to a conical pencil of rays.

Fig. 154.—Formation of Light Rays.

**Reflection of Plane Waves.**—Let AB (Fig. 155) represent the trace of a plane wave front which is perpendicular to the plane of the paper. Let this wave front be incident on the plane surface of separation of two different media, also perpendicular to the plane of the paper, and let AA′ be the section of this surface. As the incident wave travels in the direction of the arrows, it will sweep along the surface AA′, and the ether particles situated near to AA′ will be sucessively disturbed. If the properties of the media above and below AA′ are different, the conditions determining the motion of an ether particle situated just above AA′ will be affected by the presence of the lower medium. If the lower medium is denser than the upper one, the particle will be unable to move so freely as it otherwise would, since it will now have to set in motion a particle of the denser medium. If the lower medium is less dense than the upper

one, the particle will possess greater freedom than it otherwise would (p. 283). In either case, as the incident wave passes each particle along AA′, a spherical wavelet will be originated in the upper medium, which will increase in radius at a rate equal to the velocity of wave transmission in that medium. The wavelet which originated at A will have expanded to B′ by the time the incident wave has reached A′. Let A′D represent the position which the incident wave would now have occupied but for the presence of the lower medium. From A, M, and B draw the straight lines AD, MN, BA′, perpendicular to AB. Then the radius of the spherical wavelet B′ will be equal to AD or BA′, since the incident wave and the spherical wavelet travel with equal velocities. Other wavelets have meanwhile originated at points between A and A′, but as these started after that at A, they will not have travelled so far. The radius of M′, the spherical wavelet which originated at P, will be equal to PN.

FIG. 155.—Reflection of Plane Wave at Plane Surface.

A straight line through A′, perpendicular to the paper, will pass through the particles which are just being disturbed by the incident wave, and which are just about to originate spherical wavelets. Through this straight line draw a plane A′B′ tangential to the wavelet B′. Then this plane will touch all the other wavelets.

To prove this, draw AB′ and PM′ perpendicular to A′B′. Then in the right-angled triangles AB′A′ and ADA′, the sides AD and AB′ are equal, and AA′ is common to both. Hence the triangles are equal in all respects, and $\angle$ AA′D = $\angle$ AA′B′. In the two right-angled triangles PA′M′ and PA′N, $\angle$ PA′M′ = $\angle$ PA′N, the angles at M and N are right angles, and therefore the remaining angles A′PM′ and A′PN are equal. Also PA′ is the common hypotenuse to these triangles. Consequently the triangles PAM′ and PA′N are equal in all respects, and PM′ = PN. But the radius of the wavelet from P is equal to PN; thus, the wavelet from P will touch the line A′B′ at M′.

It follows that A′B′ is the section of the reflected wave front. The plane of the latter (which passes through a straight line drawn through A′ perpendicular to the plane of the paper) will obviously be perpendicular to the plane of the paper.

Since ∠ AA′B′ = ∠ AA′D, and ∠ AA′D = ∠ A′AB, the incident and reflected wave fronts are equally inclined to the surface AA′. The normals MP and M′P to the incident and reflected waves are in the plane of the paper, which also contains the normal to the surface AA′. Hence, **the normals to the incident and reflected waves, and the normal to the reflecting surface, are in the same plane.** Further, since the triangle AB′A′ is equal to the triangle ADA′, and, by construction, the triangle ADA′ is equal to the triangle A′BA, it follows that ∠ A′AB′ = ∠ AA′B, and similarly ∠ A′PM′ = ∠ APM. Thus, **the normals MP and PM′ to the incident and reflected waves are equally inclined to the surface AA′, and are consequently equally inclined to the normal to that surface.**

So far we have investigated the formation of the reflected wave fronts. We must now investigate the formation of the reflected rays. In the first place, the disturbance at A′ is due to the wavelets from the half-period zone surrounding B (p. 293), and has travelled along BA′, so that BA′ is an incident ray. Similarly MP is an incident ray. We must now prove that the disturbance at M has travelled along the line PM′, which will thus constitute the reflected ray.

Let AB (Fig. 156) be the incident wave, while A′B′ is the corresponding reflected wave. Through A describe a plane, AD, parallel to the reflected wave front B′A′. Draw the line A′D parallel to B′A, and produce M′P backwards to cut AD in Q. Then the displacements at all points in the path of the reflected wave will be exactly reproduced if we remove the lower medium, suppress the incident wave AB, and imagine that a wave front, in which the displacement is exactly equal to that in the reflected wave A′B′, starts from AD at the instant when, in reality, the incident wave passes through the position AB. For QP = MP, and DA′ = BA′; therefore the ether particles along AA′ will be disturbed in a similar succession

FIG. 156.—Incident and Reflected Rays.

by the imaginary wave AD as by the real incident wave AB. The magnitude of the disturbance of each particle due to the reflected wave will also be equal to that produced by the passage of the imaginary wave AD. Hence, we may consider the reflected wave front A′B′ to be produced by wavelets formed at all points of the imaginary wave AD. The line M′Q will be perpendicular to the plane AD, so that the disturbance at M′ is due to the wavelets originating in the half-period zone surrounding Q (p. 293). Consequently, this disturbance has travelled along QPM′, which gives the corresponding ray. The real portion of this ray, from P to M′, will be the ray reflected from P. By similar reasoning it may be proved that AB′ is the ray reflected from A.

Thus, the incident ray MP gives rise to the reflected ray PM′. Since the lines MP and PM′ are in the same plane as the normal to the surface AA′, and are equally inclined to the latter, the wave theory gives a satisfactory explanation of the laws of reflection (p. 6).

**Problems on Reflection.**—The following examples will serve to show how particular problems relating to reflection may be solved by the aid of the wave theory.

EXAMPLE 1. **Reflection of Spherical Waves at a Plane Surface.**—Let spherical waves originate at O (Fig. 157), and be reflected at the plane surface AC. Let OB be perpendicular to AC. Then B will be the first point of the surface reached by a wave DBF from O. A spherical wavelet will originate at B, at the instant when the wave DBF passes that point. Let AGC be the position which would have been occupied by the wave DBF after a short interval of time, were it not for the presence of the surface AC. In this interval the wavelet from B will have acquired a radius BH equal to BG. Wavelets are just on the point of starting

FIG. 157.—Reflection of a Spherical Wave at a Plane Surface.

from A and C, while at points between C and B, A and B, wavelets will have already started. It is evident that, these wavelets all touch a circular arc AHC, which is exactly similar to AGC. Then AHC is the trace of the reflected spherical wave on the plane of the paper. The centre of the sphere will be at O′, a point as far behind ABC as O is in front of the same. O′ is the reflected image of O in the surface ABC.

EXAMPLE 2. **Reflection of Spherical Waves at a Spherical Surface.**—Let spherical waves originate at O (Fig. 158), and be reflected at the spherical surface ABC, of which R is the centre of curvature. As an incident wave passes through the position ADC, wavelets will be formed

FIG. 158.—Reflection of a Spherical Wave at a Spherical Surface.

at A and C. Let EBF be the position which would be occupied by the incident wave, at a later period, were it not for the presence of the reflecting surface. At the instant considered a wavelet will just be starting from B, and the wavelets from A and C will have acquired radii equal to BD. Wavelets will have started from points of the surface between A and B, C and B, and these will just touch the curved surface of which GBH is the trace. This surface is the reflected wave. *If the aperture of the reflecting surface is small in comparison with the radius* CR, the reflected wave will be approximately spherical, and will converge to a point I, which is the reflected image of O.

To find the position of I, draw CG parallel to BO. Then, when the dimensions of the mirror are small, CG will be equal to the radius of the wavelet from C, or to BD. Draw CK and GL, perpendicular to BO. Let CK = GL = $y$, RB = $r$; while OB (which is approximately

equal to OD) will be equal to $u$, and IB will be equal to $v$. Then (p. 120),

$$y^2 = 2\text{BK} \cdot r = 2\text{DK} \cdot u = 2\text{BL} \cdot v.$$

But—

$$\text{BL} = \text{BK} + \text{KL} = \text{BK} + \text{CG} = \text{BK} + \text{BD}.$$

$$\therefore \frac{y^2}{2} = \text{BK} \cdot r = (\text{BK} - \text{BD})u = (\text{BK} + \text{BD})v.$$

$$\frac{y^2}{2r} = \text{BK}.$$

$$\frac{y^2}{2u} = \text{BK} - \text{BD}.$$

$$\frac{y^2}{2v} = \text{BK} + \text{BD}.$$

$$\therefore \frac{y^2}{2v} + \frac{y^2}{2u} = 2\text{BK} = \frac{2y^2}{2r}.$$

$$\therefore \frac{1}{v} + \frac{1}{u} = \frac{2}{r}.$$

This is the formula for reflection at a spherical surface which was obtained on p. 31. It may be left as an exercise for the student to find the corresponding formula, according to the wave theory, when the reflecting surface is convex.

**Reflection at a Hemispherical Surface.**—When a wave is incident on a portion of a sphèrical surface which is not small in comparison with the radius of the sphere, the reflected wave assumes a shape which it is interesting to examine. This aspect of the subject has been studied by Prof. R. W. Wood, to whom the following explanation is due.

Let ABD (Fig. 159) represent the section of a hemispherical surface by a plane passing through the centre C, and let EF be the trace of a plane wave perpendicular to the axis CB. As this wave sweeps along the surface of the hemisphere, a spherical wavelet will be generated at each point on the surface as it passes. The reflected wave surface at any instant is due to the combined action of the wavelets already formed. Let GHI (left half of Fig. 159) be the position which the incident wave would have occupied at a certain instant, but for the presence of the reflecting surface. The wavelet formed at E will by this time have acquired a radius equal to EG. With E as centre, and radius EG, describe the circular arc KL to represent this wavelet. The wavelet which started from M may similarly be represented by an arc of a circle of

radius MN. Proceeding in this manner with regard to points between E and K, we find that the various wavelets intersect along a short curve KP, which is concave upwards. Along this curve the wavelets reinforce each other, and it therefore represents the trace of part of the reflected wave surface. The wavelets which started from points between K and H similarly touch the curve PH, which represents the trace of the remainder of the wave surface. At this stage the reflected wave surface has the form of a ridge extending round the hemispherical mirror, the sharp edge of this ridge corresponding to the cusp, P, of the trace. As the incident wave moves downward, the reflected wave

FIG. 159.—Plane Wave reflected from a Hemispherical Surface.

accompanies it, the upper branch being continually replenished from the lower branch, the latter being continually re-formed by the wavelets just starting from the mirror. The right-hand half of Fig. 159 represents the construction for the trace of the reflected wave surface at an instant when, but for the reflecting surface, the incident wave would have arrived at QR. The circular arcs representing the traces of wavelets from various points of the mirror are now drawn with radii equal to the respective distances of the points from QR. The trace of the reflected wave surface has a cusp at P′, and the wave surface itself will have a form somewhat like a volcanic cone.

Fig. 160, due to Prof. Wood, shows sixteen positions of the reflected wave front. It will be seen that the concave crater

continually diminishes, until it disappears in the ninth diagram. The sides of the volcanic crater become elongated in the direction of the axis, and contracted in a direction at right angles to this, until opposite sides pass each other (ninth diagram), after which the wave surface assumes a medusa-like shape, and expands till it finally leaves the mirror.

Prof. Wood has succeeded in verifying these results with respect to sound waves in air. Fig. 161 is a reproduction of a photograph of a sound wave in twelve successive positions

FIG. 160.—Sixteen Positions of a Reflected Wave.

during reflection at a cylindrical surface. The incident wave, slightly spherical in shape (No. 1), was produced by the detonation of a small electric spark. The air in the position of the wave front was in a state of compression, and its refractive index was consequently altered. Using a modified form of Toepler's method (p. 99), instantaneous photographs were obtained, a second electric spark being used as an illuminant. The wave surface appears as a light line shaded on one side. The shape of the wave surface is similar to that predicted from the above application of Huyghens's construction.

**Caustic Curve and Focal Lines.**—In Fig. 162, a number of consecutive positions of the reflected wave front are shown in

juxtaposition. On examination it appears that the successive positions of the cusp on the trace of the reflected wave

FIG. 161.—Photographs of Reflected Sound Waves.

front lie on the caustic curve (p. 122). It is obvious from the construction used in Fig. 159 that a great number of wavelets reinforce each other in the neighbourhood of the cusp P, or P′. The cusp therefore forms a sort of moving focus, or position of maximum concentration of energy. As the cusp moves along the caustic curve, the illumination along that curve is explained.

FIG. 162.—Consecutive Positions of a Reflected Wave.

As already proved (p. 124), a reflected pencil touches the caustic curve at the position of the first focal line, and cuts the axis at the position of the second focal line for that pencil. These focal lines must be positions of maximum concentration of energy.

Let AB and BC (Fig. 163) represent the incident and reflected pencils; BC touches the caustic curve at D. But D is a particular position passed through by the cusps on the traces of all the reflected waves. Thus D is a point on the edge of the "volcanic crater" of the wave front, so that the energy focus there is an element of a ring which passes through E and D, in a plane perpendicular to the paper. Consequently the energy focus at D is a small approximately straight line perpendicular to the plane of the paper, and thus coincides with the **first focal line** of the pencil BC. During the development of the wave front the sides of the volcanic cone cross each other on the axis (p. 300). At this point the radius of the outer surface of the volcanic cone shrinks to zero. The element of the wave front corresponding to the pencil BC will cross the axis at C. The energy previously concentrated in a short line perpendicular to the plane of the paper at D will now be concentrated in a short line parallel to the axis at C; this line is the **second focal line** of the pencil BC.

Fig. 163.—Focal Lines.

**Refraction of Plane Waves.**—Let AB (Fig. 164) represent the trace of a plane wave front which is perpendicular to the plane of the paper, whilst AA′ represents the section of the plane surface of separation of two media, also perpendicular to the plane of the paper. As the wave AB (which we shall term the incident wave) travels in the direction of the arrows, the particles of the upper medium just above AA′ will be successively disturbed. Unless the lower medium is perfectly rigid, its particles just below AA′ will also be disturbed, and the disturbance of each particle will originate a spherical wavelet which will spread out in the lower medium at a rate equal to the velocity of wave propagation in that medium. The wavelets in the lower medium will generally reinforce each other along a certain surface, which is the **refracted wave front.**

At a given instant let A′D be the position which the incident wave would have occupied but for the presence of the lower medium. Draw BA′, MPN, and AD perpendicular to AB. Let $t$ be the time required for the incident wave to travel through the distance BA′ in the upper medium, and let $V_0$, $V$, be the respective velocities of wave transmission in the upper and lower media. Then, in the time, $t$, required for the incident wave to travel through the distance BA′, the wavelet which originated at A will have acquired a radius AC equal to $Vt$. Draw an imaginary line through A′ perpendicular to the plane of the paper. This line will pass through all particles which are just being disturbed, and are about to produce wavelets in the lower medium. Through this line describe a plane which touches the wavelet C. Then this plane will touch all of the wavelets which started from the points between A and A′ as the incident wave passed, and is therefore the **refracted wave front.**

FIG. 164.—Refraction of a Plane Wave at a Plane Surface.

To prove this, draw AC perpendicular to A′C. AC will be the radius of the wavelet C. Then—

$$\frac{AC}{AD} = \frac{Vt}{V_0t} = \frac{V}{V_0}.$$

Draw PM′ perpendicular to A′C. Then, since the triangles A′AC and A′PM′ are similar—

$$\frac{PM'}{AC} = \frac{A'P}{A'A}. \quad \ldots\ldots\ldots \quad (1)$$

Also, since the triangles A′AD and A′PN are similar—

$$\frac{PN}{AD} = \frac{A'P}{A'A} = \frac{PM'}{AC}, \text{ from (1).}$$

$$\therefore \frac{PM'}{PN} = \frac{AC}{AD} = \frac{V}{V_0}. \quad \ldots\ldots\ldots \quad (2)$$

But, if $r$ is the radius acquired by the wavelet which originated at P, we shall have—

$$\frac{r}{PN} = \frac{V}{V_0};$$

Therefore, from (2)—

$$r = PM',$$

and the plane AC touches this wavelet, as it does all other wavelets which originated between A and A′, as the incident wave passed.

We must now turn our attention to the incident and refracted rays. Since the disturbance at P was due to a small element of the incident wave surface, immediately surrounding M (p. 293), MP is an incident ray, and lies in the plane of the paper. Let us now suppose that the space above AA′ is filled with a medium similar to that below AA′, while the incident wave is suppressed, and the disturbance of the particles along AA′ is produced by a wave in all respects similar to A′C, which passes through A in the direction AC at the instant when the incident wave actually passes through that point. Then, by reasoning similar to that used with respect to the reflected ray (p. 295), it becomes obvious that the disturbance reaches M′ along the line PM′, so that PM′ is the refracted ray, corresponding to the incident ray MP. Now, PM′ is perpendicular to the plane A′C, which in its turn is perpendicular to the plane of the paper; therefore, PM′ lies in the plane of the paper, which also contains the normal to the surface AA′. Thus, **the incident and refracted rays, together with the normal to the refracting surface, lie in one plane.**

Let $i$ be the angle of incidence of the ray MP, while $r$ is the angle of refraction of the ray PM′. Both of these angles are measured from the normal to the surface AA′. Then, $i = \left(\frac{\pi}{2} - MPA\right) = PA'N$ $= AA'D$. Also, $r = \left(\frac{\pi}{2} - M'PA\right) = PA'M'$.

$$\left.\begin{aligned} \frac{PM'}{A'P} &= \sin PA'M' = \sin r . \\ \frac{PN}{A'P} &= \sin PA'N = \sin i . \end{aligned}\right\}$$

Therefore, from (2) above—

$$\frac{\sin i}{\sin r} = \frac{PN}{PM'} = \frac{V_0}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We thus find that **the sines of the angles of incidence and refraction bear a constant ratio to each other.** Further, when

$i$ is greater than $r$, the ratio $\sin i / \sin r$ must be greater than unity. In this case the ray is bent toward the normal during refraction, and the second medium is said to be more highly refracting than the first. Since in this case $V_0$ must be greater than V from (3), it follows that, **according to the wave theory, the velocity of light in a highly refracting medium, such as water, must be smaller than in air or in a vacuum.** This, as we have seen, was proved to be the case by Foucault (p. 228). It also follows that **the index of refraction from the first to the second medium is equal to the ratio of the velocity of light in the first medium to that in the second**; or—

$$\mu = \frac{V_0}{V}.$$

**General Construction for Reflected and Refracted Waves.**—Let AA′ (Fig. 165) be the section of the plane surface

Fig. 165.—Reflected and Refracted Waves.

of separation of two media, perpendicular to the plane of the paper. Let AB be the trace of the incident wave, its plane also being perpendicular to that of the paper. Draw BA′ perpendicular to AB, cutting the surface in A′. Let $t$ be the time required for the wave AB to travel through the distance BA′ in the upper medium, in which the velocity of wave transmission is equal to $V_0$. With A as centre, and radius

$V_0t = BA'$, describe a semicircle above AA′, and from A′ draw A′B′ tangential to this semicircle. A′B′ will be the trace of the reflected wave, the plane of the latter being perpendicular to that of the paper.

Let V be the velocity of wave transmission in the lower medium. With radius $Vt = \frac{V.BA'}{V_0} = \frac{BA'}{\mu}$, describe a semicircle below AA′, and draw A′C′ tangential to this semicircle. A′C′ is the refracted wave, its plane being perpendicular to that of the paper.

**Total Internal Reflection.**—When V is less than $V_0$, the radius AC′ of the semicircle drawn below AA′ is always less than AA′, so that the tangent A′C′ can always be drawn. Thus, when the second medium is more highly refracting than the first, there will always be a refracted wave.

When, however, the velocity of wave transmission in the lower, is greater than that in the upper, medium, it may happen that the radius AC′ is greater than the distance AA′. In this case we cannot draw a line from A′, tangential to the semicircle below AA′. There will consequently be no refracted wave, and the light will be totally reflected. The limiting case in which a refracted wave is formed occurs when the radius of the lower semicircle is equal to AA′.

Now, $BA'/AA' = \sin BAA' = \sin i$, and $BA' = V_0t$. In the case considered, $AA' = AC = Vt$, so that $\sin i = V_0/V = \mu$. Here $\mu$ is equal to the refractive index of the lower (less refracting) medium with regard to the upper one. If we let $\mu_1 = V/V_0$, so that $\mu_1$ is the refractive index of the upper (more highly refracting) medium with regard to the lower one, we have—

$$\sin i' = 1/\mu_1,$$

which determines $i'$, the limiting angle of incidence of light on the surface of the less refracting medium, in order that a refracted ray should be formed. The angle $i'$ is termed the **critical angle** for the more refracting medium.

The absence of a refracted wave, when internal reflection occurs at an angle of incidence exceeding the critical angle, is due to the circumstance that in the case considered, the wavelets formed in the rarer medium are unable to assist each other. The numerous wavelets passing through any point

differ in phase, and so produce zero displacement there. It may prove useful to examine this point a little more in detail.

Let AC (Fig. 166) be the section of the surface of separation of two media, the lower being the more highly refracting, and let AB be the trace of one of a series of plane waves incident in the direction BC on this surface from below. In order that there should be a resultant disturbance at P, a point in the upper medium, wavelets originating in the immediate neighbourhood of some point, A, must reach P in the same phase.

FIG. 166.—Illustrates Total Internal Reflection.

Let N′AN be the normal to the surface at A. Then, if $i$ is the angle of incidence, this will be the angle made by the incident rays with AN′, or that made by the wave front AB with the surface AC. Hence,

$\angle$ BAC $= i$. Let $\angle$ PAN $= \theta$. Then, $\angle$ PAC $= \dfrac{\pi}{2} - \theta$.

Let E be a point on the surface of separation very close to A. Draw ED perpendicular to the wave front. Then the light at E will be due to the combined effect of wavelets which originated in the immediate neighbourhood of D. In order that the wavelets originating at A and E should arrive at P in the same phase, the times required for light to travel from A to P, and from D through E to P, must be equal. But light travels more slowly in the lower than in the upper medium. If V and $V_0$ are the respective velocities of light in the lower and upper media, the time required for light to travel from D to E will be equal to DE/V, while the times required for light to traverse the paths AP and EP will be equal to $AP/V_0$ and $EP/V_0$ respectively. Thus, in order that wavelets from A and E should reach P in the same phase, we must have—

$$\frac{AP}{V_0} = \frac{EP}{V_0} + \frac{DE}{V}.$$

$$\therefore\ AP = EP + \frac{V_0}{V}DE = EP + \mu DE, \quad . \quad . \quad . \quad (1)$$

where $\mu$ is the index of refraction from the upper to the lower medium.

Let $AP = d$, while $AE = \delta$. Then, $DE/AE = DE/\delta = \sin EAD = \sin i$. Thus, $DE = \delta \sin i$.

Also—

$$(EP)^2 = (AP)^2 + (AE)^2 - 2AP \cdot AE \cdot \cos PAE = d^2 + \delta^2 - 2d\delta \sin \theta.$$

Then, from (1),

$$d = (d^2 + \delta^2 - 2d\delta \sin \theta)^{\frac{1}{2}} + \mu\delta \sin i.$$

$$\therefore\ d^2 + \delta^2 - 2d\delta \sin \theta = (d - \mu\delta \sin i)^2 = d^2 - 2\mu d\delta \sin i + \mu^2\delta^2 \sin^2 i.$$

$$\therefore\ \sin \theta = \mu \sin i - \frac{\delta}{2d} \cdot (\mu^2 \sin^2 i - 1). \quad . \quad . \quad . \quad (2)$$

Now, $\delta$ represents a very small magnitude, comparable with the wave-length of light. When the point P is situated at an appreciable distance from the surface AC, $d$ will be very great in comparison with $\delta$, and the second term on the right of (2) will be negligibly small. In these circumstances,

$$\sin \theta = \mu \sin i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which determines the direction AP in which the disturbance will travel in the upper medium.

This result is in accordance with the law of refraction, $\theta$ being the angle of refraction. When $\mu \sin i$ is equal to 1, $\theta = \frac{\pi}{2}$, or the disturbance will travel along the surface AC. When $\mu \sin i > 1$, it is impossible to find any value of $\theta$ which will satisfy (3). Thus, **there will be no refracted ray when the angle of incidence, $i$, is such that** $\sin i > 1/\mu$. On inspecting (2), however, it will be noticed that if $d$ is very small, so as to be comparable with $\delta$, the second term on the right may have a finite value. When $\mu \sin i > 1$, the quantity in brackets will be positive, so that the effect of the second term is to diminish the value of $\sin \theta$, and if $d$ is small enough, a real value of $\theta$ may be found from (2). The interpretation of this result is that, **for angles of incidence greater than the critical angle, there will actually be a disturbance in the rarer medium which penetrates only to a distance comparable with the wave-length of light.** We shall see, later, that there is experimental evidence of the existence of this superficial disturbance in the rarer medium.

**Refraction of Plane Waves through a Prism.**—Let AB, AC (Fig. 167) represent sections of the plane boundary surfaces of a

medium of refractive index equal to $\mu$. Let the boundary surfaces intersect in a straight line through A perpendicular to the plane of the paper, and let CD be the trace of a plane wave (also perpendicular to the paper) travelling in the direction of the arrows. To determine the wave formed by refraction at the surface AC, draw AD perpendicular to CD, and with C as centre, and radius equal to DA/$\mu$, describe a circle. From A draw AE tangential to this circle; then AE will be the trace of the wave front within the refracting medium.

FIG. 167.—Refraction of a Plane Wave through a Prism.

To find the wave front emerging from the surface AB, draw CEB perpendicular to AE, and with A as centre, and radius equal to $\mu$EB, describe a circle. Then BF, a straight line drawn from B tangential to this circle, will be the trace of the emergent wave front. If AF is drawn perpendicular to BF, AF will be the emergent ray corresponding to the incident ray DA.

It should be noticed that the path of the end D of the wave is wholly in air, while that of the end C is wholly in the refracting medium. The path DA + AF is $\mu$ times as long as the path CB, so that disturbances starting simultaneously from D and C, will arrive in the plane BF at the same instant.

The **deviation** produced by refraction through the prism may be measured either by the angle between the incident and emergent rays, or by that between the incident and emergent waves. Thus, if DC and FB are produced to meet in G, the angle DGF will give the deviation produced. For this deviation to be a *minimum*, it can be proved that DC and FB must be equally inclined to the respective faces AC and AB, in which case it is easily seen that AE will bisect the angle CAB.

The existence of an **angle of minimum deviation** can be proved by a very simple geometrical construction. Let AB, AC (Fig. 168) be sections of the plane faces of the prism. Let AD be drawn parallel to the trace of the wave incident on the face AC. Then $\angle$ DAC will be the angle of incidence on the face AC. With A as centre, and with radii in a ratio equal to the refractive index of the medium composing the prism, describe the circular arcs BC and RS. From E, the point of intersection of AD with the arc RS, draw EP parallel to AC, intersecting the arc BC in P. Join AP. Then AP will be parallel to the wave inside the prism. To prove this, draw PM and EN perpendicular to AC. Then PM = EN, and, by construction, AP = $\mu \times$ AE. Consequently, sin EAN = EN/AE = $\mu \times$ (PM/AP) = $\mu$ sin PAM. Thus, PAM is the angle of refraction corresponding to the angle of incidence EAN, or PA is the refracted wave corresponding to the incident wave AD.

To determine the emergent wave, draw PF parallel to AB, and intersecting the arc RS in F. Join AF. Then AF is parallel to the emergent wave. To prove this, draw PK and FL perpendicular to AB. Then FL = PK, and sin FAL = FL/AF = $\mu$(PK/AP) . = $\mu$ sin PAK. But PAK is the angle of incidence of the wave AP on the face AB. Since the wave AP is travelling through a medium of refractive index equal to $\mu$, and is incident on the surface of separation of that medium from one in which the refractive index is equal to unity, we must have—

FIG. 168.—Angle of Minimum Deviation.

$$\text{sin angle of incidence} = \frac{1}{\mu}\,\text{sin angle of refraction.}$$

$\therefore$ $\mu$ sin PAK = sin angle of refraction = sin FAL.

$\therefore$ FAL is the angle of refraction, and AF is parallel to the emergent wave.

Since AD and AF are respectively parallel to the incident and emergent waves, the deviation produced is equal to the angle DAF. This angle is measured by the arc EF, cut off by the two lines PE and PF from the arc RS. Therefore, the deviation will have a *minimum*

value when the arc EF has the *smallest possible* value. Now, by altering the position of the incident wave AD, we may alter the position of P on the arc BC, but PF is always parallel to AB, and PE to AC. It can easily be seen that, in these circumstances, the lines PF and PE will cut off a minimum length from RS when P bisects the arc BC. The wave AP inside the prism will then bisect the angle CAB, and the incident and emergent waves will be equally inclined to BA and CA respectively.

**Mechanical Illustration of Refraction.**—We have seen that the refraction of a light-wave at the surface of a refracting medium is due to the circumstance that the end of the wave which first enters the refracting medium is retarded, and the whole wave is consequently forced to swing round through a certain angle. The following mechanical arrangement illustrates this point.

Two boxwood wheels, about two inches in diameter, with rounded rims, are mounted on the ends of an iron axle, about four inches in length and one-half inch in diameter, so as to be able to rotate freely and independently of each other (Fig. 169). If this arrangement is allowed to roll down a slightly inclined wooden board, it will pursue a straight course. If part of the board is covered with thick pile plush (termed artificial sealskin), then the above arrangement will travel more slowly against the direction of the pile than it would on the plain board. If a parallel strip of this plush is glued obliquely across the board, with the direction of the pile sloping upwards, then the wheel which first comes on the plush will be retarded in its course, and the unretarded motion of the other wheel will cause the axle to swing round till both wheels roll on the plush, after which the course will be straight but inclined to its previous direction. On leaving the plush the course will once more assume its original direction (Fig. 170). This represents the refraction of a light-wave through a parallel

FIG. 169.—Roller used in illustrating Refraction.

FIG. 170.—Mechanical Illustration of Refraction.

plate of glass. Using a triangular piece of plush, the refraction of a wave by a prism may be illustrated (Fig. 170). If a piece of plush is cut in the shape of an isosceles triangle with a vertical angle equal to 90°, then the total internal reflection of light may be illustrated (Fig. 171). The wheel which first leaves the plush moves so much more quickly than the one still on it, that the axle swings round, and the first wheel once more arrives on the plush before the other one has left it. This illustrates in a very striking manner the creation of only a superficial disturbance in the less refracting medium when the angle of incidence exceeds its critical value (p. 306).

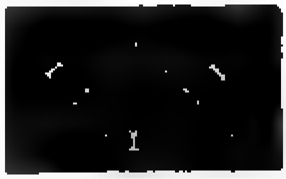

FIG. 171. — Mechanical Illustration of Total Internal Reflection.

**Problems on Refraction.**—The following examples will serve to illustrate the solution of problems on refraction by the aid of the wave theory.

EXAMPLE 1. **Refraction of a Spherical Wave at a Plane Surface.**—Let BAC (Fig. 172) represent the trace of a spherical wave diverging

FIG. 172.—Refraction of a Spherical Wave at a Plane Surface.

from O, and just touching, at A, the plane surface DAE of a refracting medium. A spherical wavelet originates at A and spreads out into the refracting medium at a rate equal to the velocity of light in

that medium. Let DFE be the position which the incident wave would have occupied after a short interval of time, $t$, but for the presence of the refracting surface. Then, if $V_0$ and V denote the respective velocities of wave transmission in the media to the right and left of DAE, the distance AF will be equal to $V_0t$, and the radius AG of the spherical wavelet which originated at A will be equal to $Vt$. Thus, $AF/AG = V_0t/Vt = \mu$, where $\mu$ is the refractive index of the medium to the left of DAE, with respect to that on the right of DAE. Spherical wavelets will be on the point of starting from D and E, and at points between A and E, or A and D, wavelets will have already started, but will have acquired radii less than AG. The curve DGE which touches all of the wavelets will be the trace of the refracted wave front. This curve is not a circle, so that the refracted wave will not be spherical, and will not therefore diverge from a point. But if the aperture AE of the surface exposed to the waves is small in comparison with AO, the refracted wave will be approximately spherical, and will diverge from a point I, which is the image of O.

Let OA (which is approximately equal to OF) be equal to $u$, while IA (which is approximately equal to IG) is equal to $v$. Then, if $AE = y$, we have—

$$y^2 = 2AF.u = 2AG . v.$$

But $AF = \mu AG$.

$$\therefore \mu u = v.$$

This is the result already obtained and discussed (p. 53).

EXAMPLE 2. **Refraction of a Spherical Wave at a Spherical Surface.**—Let BFC (Fig. 173) be the trace of a spherical wave diverging from O, and let BAC be the section of a spherical surface with centre at R, the velocities of wave transmission in the media to the right and left of BAC being respectively equal to $V_0$ and V. Let $t$ be the time required for the point F on the incident wave to travel to A; then $AF = V_0t$. In the time $t$ the wavelets which originated at B and C will have acquired radii equal to $Vt$, or $(AF . V/V_0)$. The wavelets which originated between A and C, or A and B, will have acquired smaller radii, the wavelet from A being just on the point of starting. The curve DAE, which just touches all of the wavelets, will be the refracted wave front. If the aperture AC of the refracting surface is small, the curve DAE will be approximately spherical, and will diverge from a point I. Then I is the image of O.

Draw CE parallel to the axis OA. Then CE will be approximately equal to the radius of the wavelet which originated at C, or to $AF/\mu$. Draw EG, and CH perpendicular to OA, and let $y = GE = HC$. Then,

if AR = $r$, while AO (which is approximately equal to FO) = $u$, and AI = $v$, we have—

$$y^2 = 2\text{AH} \,.\, r = 2\text{FH} \,.\, u = 2\text{AG} \,.\, v.$$

Then, since AG = AH − GH = AH − EC = AH − $\dfrac{\text{AF}}{\mu}$, we have—

$$\text{AH} - \frac{\text{AF}}{\mu} = \frac{y^2}{2v}. \qquad \qquad (1)$$

Also, since FH = AH − AF,

$$\text{AH} - \text{AF} = \frac{y^2}{2u}. \qquad \qquad (2)$$

Multiply (1) throughout by $\mu$, and from the result subtract (2). Then—

$$(\mu - 1)\text{AH} = \frac{y^2}{2}\left(\frac{\mu}{v} - \frac{1}{u}\right).$$

Then, since AH = $y^2/2r$,

$$\frac{(\mu - 1)y^2}{2r} = \frac{y^2}{2}\left(\frac{\mu}{v} - \frac{1}{u}\right); \quad \therefore \frac{\mu}{v} - \frac{1}{u} = \frac{(\mu - 1)}{r}.$$

FIG. 173.—Refraction of a Spherical Wave at a Concave Surface.

It may be left as an exercise to the student to obtain, according to the wave theory, the corresponding formula when the surface is convex.

EXAMPLE 3. **Refraction through a Lens.**—The result of Example 2 can be used, as on p. 67, to find the formula for refraction through a lens. As, however, the direct solution of this problem is very instructive, a few lines will here be devoted to it.

Let BAC (Fig. 174) be the trace of a spherical wave diverging from O, a point on the axis of the glass lens KL. In order that a real image may be formed by refraction through the lens, the emergent wave EDF must be spherical, and converge toward a point I on the axis. If the surfaces of the lens are spherical (as is almost universally the case) the emergent wave will be approximately spherical only when the aperture, AL, of the lens is small in comparison with AO and DI. Assuming this to be the case, it becomes obvious that the action of the

FIG. 174.—Refraction through a Lens.

lens is to retard the central part of the wave with respect to its peripheral portions, and so alter its curvature. The disturbance from C must traverse the path CLF in air, in the time required for the disturbance from A to reach D, travelling through the glass. Join CF. Then, in the case considered, CF will be approximately parallel to the axis, and will be equal in length (to a first approximation) to the air path CLF. Draw CG, LM, and FH perpendicular to the axis. The line FC will be approximately equal to HG.

Let $V_0$ and V be the velocities of light in air and glass respectively, so that the refractive index of the glass is equal to $\mu = V_0/V$. The time required for light to travel from C to F in air $= CF/V_0 = GH/V_0 = (GA + AD + DH)V_0$. The time required for light to travel from A to D in glass $= AD/V$. Then—

$$(GA + AD + DH) = AD \cdot \frac{V_0}{V}.$$

$$\therefore \; GA + DH = \left(\frac{V_0}{V} - 1\right) AD = AD \cdot (\mu - 1).$$

Let $r_1$ and $r_2$ be the radii of curvature of the surfaces KAL and

KDL respectively. $r_1$ will be negative (p. 59). Let AO $= u$, while DI $= v$; $v$ will be negative. Then, if GC $=$ ML $=$ HF $= y$ (to a first approximation) we shall have—

$$GA = \frac{y^2}{2u}, \; AM = -\frac{y^2}{2r_1}, \; MD = \frac{y^2}{2r_2}, \text{ and } DH = -\frac{y^2}{2v}.$$

$$\therefore \; GA + DH = \frac{y^2}{2}\left(\frac{1}{u} - \frac{1}{v}\right), \text{ and } AD = AM + MD = \frac{y^2}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

$$\text{Thus, } \frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \text{ the result obtained on p. 68.}$$

The student should find no difficulty in obtaining the corresponding formula for a divergent lens. It will be noticed that in this latter case the peripheral portions of the incident wave are retarded with respect to the central portions.

**Dispersion.**—When a parallel pencil of white light is refracted through a transparent prism, and then allowed to fall on a sheet of white paper, it is observed that the illumination produced is not uniform ; the luminous patch seen is brightly coloured, and experiments already described show that the emergent light consists of an indefinite number of parallel pencils of coloured light, each being deviated by an amount depending on its colour. This variation in the refraction of light by a material medium, dependent on the colour of the light, is termed **dispersion**. The pencils producing a red coloration are deviated least, while those producing a blue or violet coloration are deviated most. This result suggests that white light consists of an indefinitely large assemblage of waves which are not identical but may be distinguished from each other by some characteristic property which, acting on the eye, produces the sensation of colour.

Now the nature of a wave becomes known when its amplitude, its period, and the velocity with which it is propagated are known. The wave-length, $\lambda$, and the period, T, of a wave are connected with the velocity of propagation, V, by the relation

$$\lambda = VT, \text{ (p. 261).}$$

We have ample evidence that the velocity of light in a vacuum is independent of the colour of the light (p. 226). The amplitude of a wave determines the intensity or brightness of

the light. It therefore appears probable that the phenomena of coloration are due to differences of period in the light-waves. Thus, light corresponding to a given part of the spectrum is probably characterised by a definite period, or a definite wave-length when travelling through a vacuum.

For waves, initially travelling with equal velocities, to be deviated by different amounts when transmitted through a prism, their velocities within the prism must be different. The waves corresponding to blue or violet light must travel the most slowly within the prism, since they are deviated most, while waves corresponding to red light must travel most quickly, and those corresponding to intermediate portions of the spectrum must travel with intermediate velocities. It remains to be determined whether the blue or the red light-waves possess the greater wave-length.

**Interference.**—Let A and B (Fig. 175) be two points on the surface of a liquid, which are subjected to harmonic displacements equal in amplitude and phase. Circular waves will spread out from A and B, and the amplitudes of the two wave trains will be equal. At a particular instant the displacements at any point on the surface will be the resultant of the displacements due to the two waves then passing through the point. In Fig. 175, the continuous circular arcs represent the crests, while the broken arcs represent the troughs of the waves diverging from A and B. Through the points, marked by crosses, two crests or two troughs are simultaneously passing, so

FIG. 175.—Illustrates the Interference of Waves.

that at these points the waves reinforce each other, and the resultant displacement is equal to twice the amplitude of either of the passing waves. Since the energy at any point varies as the square of the amplitude (p. 276), the energy at a point marked by a cross will be equal to four times the energy due to a single wave. The lines joining the neighbouring points marked by crosses will always be lines of maximum disturbance and maximum energy; for, after a time equal to half the period of the waves, the only change that will have occurred is that a point, at which the displacement was previously produced by the combined action of two crests, will now be subjected to a displacement due to two troughs.

Through each of the points marked by small circles a trough and a crest are simultaneously passing. At these points the resultant displacement will be equal to zero, since the component displacements are equal in magnitude but opposite in direction. The lines joining neighbouring points marked by small circles will always indicate portions of the surface which are stationary; for after a time equal to half the period of the waves the displacement at any point on one of these lines will be due (say) to a crest and a trough, instead of a trough and a crest. Along the lines of zero displacement the waves are said to **interfere** with each other.

It must be clearly understood that interference can never produce a loss of energy. The energy missing from the stationary points on the surface is merely transferred to the points of maximum displacement.

Fig. 176 is a reproduction of an instantaneous photograph of ripples on the surface of mercury, obtained by Dr. J. H. Vincent.[1] The ripples were produced by two glass styles attached to the same prong of a vibrating tuning-fork. Thus, the harmonic motions of the styles were necessarily in the same phase, and the circular waves started from the disturbed points on the mercury surface in the same phase. The unshaded lines, radiating from a point midway between the two wave sources, mark the lines of zero displacement.

Returning now to Fig. 175, it is readily seen that if the velocity is the same for waves of all periods, then the angular

[1] "On the Photography of Ripples," Dr. J. H. Vincent, *Proc. Phys. Soc.*, vol. 15, p. 91.

distance between neighbouring lines of maximum displacement will be proportional to the period, or to the length, of the waves. For, if we wished to change Fig. 175 so as to represent the effect of waves of half the length, we should have to substitute continuous lines for the broken lines, since the distance between two crests (one wave-length) is now equal to what was previously the distance between a crest and a trough (half a wave-length). Thus, the points in Fig. 175 marked by small circles would now have to be marked with crosses, indicating points of maximum displacement. In addition, midway between each pair of circular arcs already drawn we should have to describe broken circular arcs to represent the new troughs of the waves, so that we should obtain a new set of lines of zero displacements, these lines being twice as numerous as previously.

Vincent.)

**Interference of Light-Waves.**—If light consists of waves, it ought to be possible to obtain effects due to interference. To produce these effects we should, in the first place, need two sources continuously emitting waves *of the same period and phase.* We could not hope to produce interference between the lights emitted, say, by two separate candles, since in that case there would be no relation between the phases of the waves

produced. In the second place, if the wave-length of light is very small, it would be necessary to place the wave sources very near to each other for the angular distances between lines of maximum displacement to be appreciable. This condition can easily be understood if we remember that the distance between C and D (Fig. 175) must be such that the distance from B to D is only half a wave-length longer than that from A to D. Lastly, we cannot directly observe the motions of the ether, but only the resultant luminous effects produced at a point by the passage of billions of waves in a single second.

Let us suppose that A and B (Fig. 175) are sections of two linear sources of light perpendicular to the plane of the paper. If waves of equal amplitude simultaneously originate at these sources in the same phase, the circular arcs will represent the traces of crests and troughs of cylindrical waves. The imaginary lines joining neighbouring points marked by small circles will represent the traces of surfaces approximately plane, and perpendicular to the plane of the paper, characterised by the peculiarity that in them the ether is permanently at rest. In the space between two neighbouring surfaces of zero displacement (such as those passing through D and E), waves will travel outward, and will illuminate any obstacle placed in their path. Let FL be the section of a plane white screen perpendicular to the plane of the paper. Then the points G and K, being situated in surfaces of zero displacement of the ether, will be unilluminated. Thus, there will be two black bands perpendicular to the plane of the paper, through the points G and K on the screen. The point H, midway between these bands, will be brightly illuminated, since waves will travel outward to that point between the imaginary planes of zero displacement. Thus, there will be a bright band, perpendicular to the plane of the paper, through the point H on the screen. For similar reasons there will be bright bands through the points F and L on the screen, and if the construction used in Fig. 175 were extended, we should find that in passing from H outwards through K and L, alternate dark and bright bands would be encountered on the screen (compare Fig. 179). These bands are termed **interference bands or fringes.** Let us now determine in what manner the breadth of the interference fringes will depend on the length of the light-waves. The point H is equidistant from A and B;

thus similar waves starting simultaneously in the same phase from A and B will reinforce each other at H, whatever may be their length. The bright band through H is termed the central interference fringe. The first dark band will occur at a point, K, such that the distance from K to B is one half wave-length greater than that from K to A. If we diminish the length of the waves, the distance HK will be diminished, so that the breadth of the central band will depend on the length of the light-waves. At the point L, the wave from B arrives one whole period later than that from A, so that the distance from L to B is one wave-length longer than that from L to A. If we diminish the wave-length, we shall diminish the distance HL from the middle of the central fringe to the middle of the first succeeding bright fringe. Reasoning in this manner, it is plain that the central interference fringe will have the same position whatever may be the length of the light-waves, but the fringes formed by the interference of short waves will be narrower than those formed by the interference of longer waves.

Fig. 177 represents the nature of the interference fringes which would be produced by red and blue light waves, if the wave-length is greater for the red than for blue waves. If the sources A, B, (Fig. 175), simultaneously emit red and blue waves, these two sets of interference fringes would be superposed. The central band would be coloured by a mixture of red and blue light, but its edges would be red, since the width of the blue fringe is less than that of the red one. On the other hand, the inner edge of the first interference fringe would be blue, since at that point red light is absent. Passing outwards, it is obvious that instead of a dark band following on the first interference fringe, we shall have a blue band, since the second blue interference fringe coincides with the dark band between the first and second red fringes.

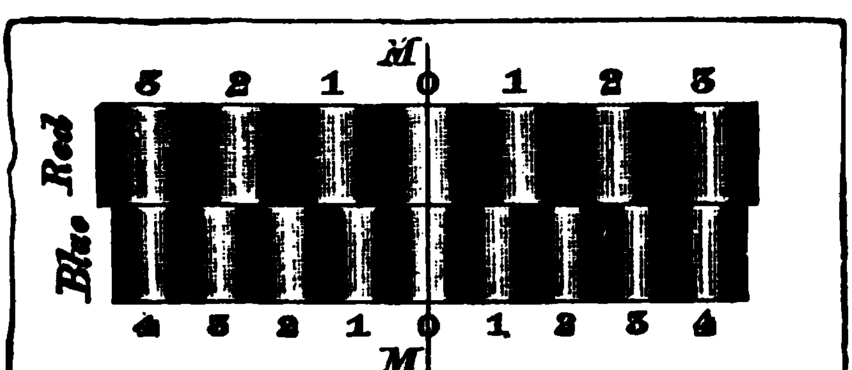

FIG. 177.—Nature of Interference Fringes, respectively produced by Red and Blue Light Waves.

Let us now suppose that white light is composed of waves of various lengths, decreasing regularly from the red to the blue end of the spectrum. It is plain that if the sources A and B (Fig. 175) emit such light, the central fringe will be formed by the superposition of the bright bands of all wave-lengths; it will therefore be white at the middle, shading off into red at its edges. The next bright band will be blue on its inner edge, and will be coloured at its middle, since here the bright bands due to the various wave-lengths will not coincide. Thus, starting from the central white fringe, we shall encounter a number of brilliantly coloured bands which are, generally speaking, bluish on their inner, and reddish on their outer, edges. At a short distance from the central fringe the coloured bands will become dim, and ultimately disappear, due to the superposition of a great number of different fringes producing uniform illumination.

**Fresnel's Interference Experiment.**—Fresnel was the first investigator to produce effects incontestably due to the inter-

FIG. 178.—Fresnel's Double Mirror Interference Experiment.

ference of light. The arrangement he used is represented diagrammatically in Fig. 178. A narrow slit, S, perpendicular to the plane of the paper, was illuminated by sunlight, and the light issuing from it was reflected from the plane mirrors MO and ON, which were very nearly parallel to each other, but intersected in a straight line through O perpendicular to the paper. The light reflected from MO appeared to proceed from A, the image of S in MO. Similarly, the light reflected from

ON appeared to proceed from B, the image of S in ON. By altering the inclination of the mirrors, the distance between A and B could be adjusted at pleasure. Thus A and B were the virtual sources of light, and being images of S, the reflected waves virtually originated at A and B in the same phase. The point C on the screen was equidistant from A and B, and was found to lie on a white fringe bordered with red. On either side of C were brilliantly coloured bands (Fig. 179), which, generally speaking, were bluish on their inner, and reddish on their outer, edges. That these bands were actually produced by interference was proved by covering up one of the mirrors, when all traces of the fringes disappeared. By altering the inclination of the mirrors, it was found that the width of the fringes increased as the images A and B were brought close to each other, which is the result to be anticipated from theory. When the slit S was covered with a piece of red glass, the bands were alternately red and black; the width of the fringes thus produced was greater than when the slit was illuminated by blue or green light. This proves that the waves corresponding to the blue, are shorter than those corresponding to the red, portion of the spectrum. When the slit was illuminated by white light and the screen was viewed through a piece of red glass, the bands seen were alternately red and black, as in the case when the slit was illuminated by red light.

FIG. 179.—Fresnel's Double Mirror Interference Fringes. (From a photograph by Prof. Chant.)

**Fresnel's experiment gives decisive evidence in favour of the wave theory of light.** That light when added to light should produce darkness is incomprehensible on any theory of the material nature of light. In addition, **Fresnel's experiment proved that white light consists of numerous waves of which the length decreases from the red to the violet end of the spectrum.**

**Polarisation of Light.**—Having decided that light consists of waves propagated through the ether, it remains to determine the nature of these waves. Are they, for instance, purely compressional waves, such as the sound-waves transmitted through a gas? Or, if the ether possesses shear elasticity (p. 267), are they waves of longitudinal displacement (p. 268), or of transverse displacement (p. 269)? To these questions we are able to give a decided answer.

Imagine a string passing at right-angles through a slit in a diaphragm. Longitudinal vibrations, consisting of backward and forward motions transmitted along the string, cannot be affected by the orientation of the slit. On the other hand, transverse vibrations can only be transmitted through the slit, if they are performed parallel to it. Thus, if the slit is arranged so that transverse vibrations in a certain plane are transmitted along the string, rotating the diaphragm in its own plane till the slit is at right angles to its previous position, will prevent their further transmission.

On looking at a sheet of white paper through a crystal of tourmaline cut parallel to its axis, nothing remarkable is noticed; the light is slightly coloured, due to the natural colour of the crystal, and that is all that our eyes can tell us. If we place two similar crystals face to face with their axes parallel (A, B, Fig. 180), the only observable difference produced is an increased colora-

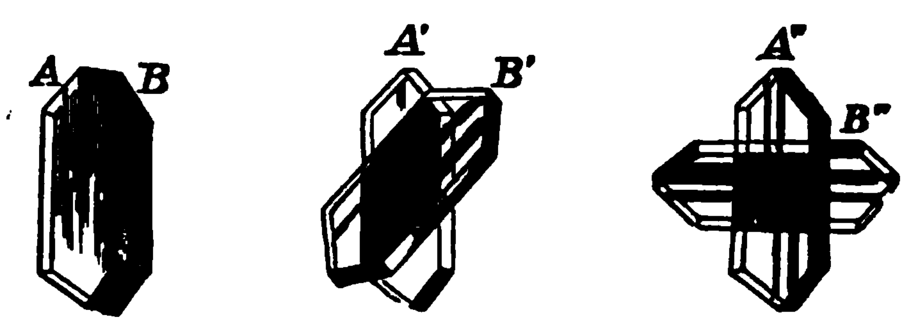


FIG. 180.—Illustrates the Extinction of Light by Crossed Tourmalines.

tion of the emergent light. Rotating both crystals together in a plane parallel to their faces produces no difference. If, on the other hand, we rotate one crystal with respect to the other, the light transmitted through the two becomes dimmer and dimmer (A′, B′, Fig. 180), until total extinction occurs when the axes of the crystals are at right angles (A″, B″, Fig. 180). As the angle between the axes is further increased, more and more light is transmitted, until on completing a rotation through 180° the same amount of light emerges as in the original position.

**This experiment proves that light does not consist of compressional or longitudinal waves**, for it is inconceivable that a rotation of the second crystal about the direction of the ray, and therefore about the direction of vibration (if longitudinal), should extinguish the light. After passing through the first crystal the light-waves have acquired a one-sidedness with regard to the direction of the ray. It is now said to be **polarised. We are therefore forced to conclude that the direction of displacement in a light-wave is perpendicular to the direction of transmission.** In unpolarised light we may suppose that the actual direction of displacement changes a great many times a second, always, however, remaining perpendicular to the direction of transmission. The first tourmaline crystal only transmits vibrations making a certain angle with its axis. If the axis of the second crystal is parallel to that of the first, the waves transmitted through the first crystal can traverse the second also. If, however, the axes of the crystals are at right angles to each other, the light transmitted through the first crystal consists of vibrations in a direction at right angles to that in which alone they could be transmitted through the second crystal. Thus, the two crystals with axes at right angles allow no light to pass.

Imagine a long string passing, at two points in its length, through slits in different diaphragms. If one end of the string is caused to move transversely in various directions, transverse vibrations in various directions will travel along the string. The first slit will only allow vibrations parallel to its length to pass, and these vibrations will be unable to pass the second slit if it is perpendicular to the first one.

**The Nature of the Ether.**—Assuming interstellar space to be filled with a continuous medium, the ether, which can transmit waves, the arrival, at the earth, of light from the stars can be explained. The velocity of wave transmission would be equal to the square root of the ratio of the elasticity to the density of the ether, and would be the same for waves of all lengths (p. 274). This accounts for the fact that light travels with one uniform velocity through a vacuum, whatever may be its colour. The reflection of light at a material surface will follow naturally, if matter modifies either the elasticity or density of the ether. If the velocity of wave transmission through a transparent material medium is less than that

corresponding to interstellar space, the refraction of light is explained. We may enquire whether light is transmitted through a transparent material medium in the same way as, for instance, sound is transmitted through a solid or liquid ; that is, by means of vibrations confined to the material medium. This question must be answered in the negative. The square root of the ratio of the elasticity to the density of any material substance is many thousand times smaller than the velocity of light. The density of crown glass is equal to 2·5, while its rigidity is equal to $1·5 \times 10^{11}$, and its compressional elasticity is equal to $4·2 \times 10^{11}$. Thus, the velocity of longitudinal waves in glass is equal (pp. 267–9) to $\sqrt{\dfrac{4·2 \times 10^{11} + \frac{4}{3} \times 1·5 \times 10^{11}}{2·5}}$ $= 5 \times 10^5$ cms. per second. Transverse waves would be transmitted through glass with a velocity of $\sqrt{\dfrac{1·5}{2·5} \times 10^{11}}$ $= 2·8 \times 10^5$ cms. per second (p. 273). On the other hand, taking 1·5 as the mean refractive index of crown glass, light must be transmitted through it with a velocity equal to $\dfrac{3 \times 10^{10}}{1·5} = 2 \times 10^{10}$ cms. per second. Thus, whatever may be the nature of light-waves, their velocity of transmission through glass is far greater than that corresponding to waves transmitted merely through the glass itself. As a consequence we must assume that material media are penetrated by the ether, their molecules being surrounded by it much as the leaves of a tree are surrounded by the air. When light traverses a material substance, it is transmitted by the ether penetrating that substance ; the molecules of the substance in some manner modify the properties of the ether immediately surrounding them, so as to diminish the velocity of light, to an extent dependent on the period or length of the transmitted waves. We may suppose that the elasticity of the ether remains unaltered, while its effective density is increased by the reactions of the molecules of a material substance (p. 283). This is the meaning of the statement that glass is an optically denser medium than air. As a general rule, a high optical density goes with a high mechanical density (mass per unit volume) of a substance. However, the velocity of light in a medium is not invariably

connected with the mechanical density of the latter. Thus, oil of turpentine, which floats on water, has a mean refractive index equal to 1·46, while that of water is equal to 1·33.

The phenomena of polarisation force us to conclude that the displacement in light-waves is transverse to the direction of transmission. Accordingly we must assume that the ether is endowed with properties which enable it to transmit waves of transverse displacement. In other words, the ether must possess properties similar to those of an elastic solid, such as a jelly. An ordinary jelly possesses two kinds of elasticity: one, by which it resists compression, is usually very great; the other, by which it resists change of shape, or distortion, is of much smaller magnitude. We have no evidence of longitudinal or compressional waves in the ether, and, to account for this, it is generally assumed that the ether is incompressible. It will then merely be able to transmit transverse waves, their velocity being equal to $\sqrt{(\eta/\rho)}$ (p. 273).

A serious difficulty arises at this point. It is difficult to imagine the planets as moving with their enormous velocities through a jelly-like substance without any loss of energy. The motions of the planets are perfectly regular, and show no signs of any loss of this kind. It is true that Encke's comet has been observed to return to its perihelion position a little before the calculated time, and this has led to the supposition that its motion is retarded by the ether. The comet describes an elliptic orbit of great eccentricity; if its velocity is diminishing, it will travel to less and less distances from the sun during successive revolutions, and its time of revolution will thus slowly decrease. It would be rash, however, to found a theory on this isolated observation. There is, then, no certain evidence of the continuous motion of a body being resisted by the ether. The difficulty of reconciling the absence of such resistance with the properties of an elastic solid has always been a source of difficulty in relation to the wave theory; it prevented Fresnel from publishing his conclusions in this respect until after Dr. Young had propounded the same theory, which he had arrived at independently.

These difficulties are greatly diminished if we remember that although the elasticity must bear a great ratio to the density of the ether, both of these quantities may be very small. Lord

Kelvin[1] has recently estimated that the density of the ether is of the order of $5 \times 10^{-18}$, the density of water being equal to unity. If this were so, we should have $\eta/(5 \times 10^{-18}) = V_0^2 = (3 \times 10^{10})^2 = 9 \times 10^{20}$. Thus $\eta = \frac{9}{5} \times 10^2 = 180$, which is a very small elasticity, much less than that pertaining to a weak solution of glue in water, which would, as far as the motion of a solid through it is concerned, be quite fluid.

In this connection the following opinion of Sir George Stokes[2] should carry much weight :—

"The supposition that the ether would resist . . . a body moving through it is derived from what we observe in the case of solids moving through fluids, liquid or gaseous, as the case may be. In ordinary cases of resistance, the main representative of the work apparently lost in propelling the solid is in the first instance the molecular kinetic energy of the trail of eddies in the wake. The formation of these eddies is, however, an indirect effect of the internal friction, or—if we prefer the term—viscosity of the fluid. Now the viscosity of gases has been explained on the kinetic theory of gases, and in the case of a liquid we cannot well doubt that it is connected with the constitution of the substance as not being absolutely continuous but molecular. But if the ether be either non-molecular, or molecular in some totally different sense from ponderable matter, we cannot with safety infer that the motion of a solid through it necessarily implies resistance."

## Questions on Chapter XIII

1. Write an essay on: The principle of interference as applied to explain the rectilinear propagation of light.

2. Explain the refraction of light by a plane surface according to the wave theory.

3. Apply the undulatory theory to determine the path of a ray of light through a prism, and show from your construction that the deviation is least when the angles of incidence and emergence are equal.

[1] "Ether and Gravitational Matter in Space," Lord Kelvin, *Phil. Mag.*, August 1901, pp. 161–177.

[2] Presidential Address at Anniversary Meeting at Victoria Institute, June 29, 1893, *Nature*, July 27, 1893.

4. Apply the principles of the wave theory to account for the formation of a real image by a convex lens, and by the aid of your explanation deduce the formula $\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$.

5. What do you understand by the interference of light? Why is the principal focus of a convex lens, when placed in a pencil of parallel rays, a point of maximum brightness?

6. Give a general explanation of the interference bands produced in white light by the use of Fresnel's mirrors.

7. What evidence is there that ordinary light consists of trains of waves, each train consisting of many successive waves? If waves of light could be produced having only one or two waves in each train, how would the phenomena of reflection, refraction, diffraction, and polarisation be modified?

# CHAPTER XIV

## THE SPECTRUM AND ITS TEACHINGS

**Line Spectra.**—Since light-waves of different lengths are unequally deviated when refracted through a transparent prism, we have a means of studying the nature of the waves composing any particular kind of light. The spectrometer and its adjustment have already been described. The light entering the slit of the instrument is rendered parallel by the collimating lens, and after being refracted through the prism, is focussed by the objective of the telescope, and the real image formed is viewed by the aid of the eye-piece. If the slit is illuminated by monochromatic light (*i.e.*, light of only one wave-length), then the parallel pencil leaving the collimator is refracted through the prism, and leaves the latter as a single parallel pencil which is brought to a focus by the objective of the telescope. In this case a single image of the slit is formed in the focal plane of the objective, and this image, viewed through the eye-piece, is seen as a vertical luminous line of a definite colour. When common salt is introduced into the non-luminous flame of a Bunsen burner, an intensely yellow light is emitted. If this light is used to illuminate the slit of a spectrometer, the image seen in the focal plane of the telescope consists of two bright yellow lines (termed the D lines) separated by a dark interval depending on the dispersive power of the prism, and the magnifying power of the telescope. Thus, the light emitted by incandescent sodium vapour, when analysed by a spectrometer, produces two images which must correspond to two different wave-lengths.

When other metallic salts are introduced into a Bunsen flame, definite flame colorations are generally produced, and

examination by means of a spectrometer shows that in each case light-waves of definite lengths are emitted. This result is of capital importance, for it gives us a means of detecting the presence of various metallic substances, even when present in small traces, in a mixture. It has also led to the discovery of elements previously unknown. For, if the spectrum of a substance contains lines which do not correspond to those of any known element, the obvious conclusion is that the substance contains an element heretofore unknown. In this manner Bunsen discovered the elements *cæsium* and *rubidium*, Crookes discovered *thallium*, and Reich and Richter discovered *indium*.

The exact wave-lengths of the radiations emitted by various substances have been determined by means which will be fully described later. By the aid of these results we may determine the wave-length corresponding to any unknown spectral line. Having adjusted a spectrometer, we may determine the deviations corresponding to waves of known lengths, emitted by various metallic salts, and then draw a curve, with deviation as abscissæ, and wave-lengths as ordinates. By the aid of this curve we can determine the wave-length of any unknown line. All we have to do is to observe the deviation of the line in question, when the wave-length corresponding to this deviation may be directly read off from the curve.

EXPT. 55.—Calibrate a spectrometer so as to determine the relation between deviation and wave-length for different parts of the spectrum. The spectrometer must be adjusted in the usual manner (p. 88), various salts being introduced into the Bunsen flame used as an illuminant. The following table gives the wave-lengths of the luminous radiations emitted by a number of metals. A small amount of the salt of a metal may be introduced into the Bunsen flame on a spiral of platinum wire. In most cases the light emitted is brighter if the salt is moistened with strong hydrochloric acid.

The following units are used in the measurement of wave-lengths. A **tenth-metre** is equal to $10^{-10}$ metre. A **micron** ($1\ \mu$) is equal to a thousandth part of a millimetre ($10^{-3}$ mm., or $10^{-4}$ cm.). A **micro-millimetre** ($1\ \mu\mu$) is equal to a thousandth part of a micron, or a millionth part of a millimetre ($10^{-6}$ mm., or $10^{-7}$ cm.). Thus, 1 tenth-metre $= 10^{-10}$ metre $= 10^{-8}$ cm. $= 10^{-7}$ mm. $= 0{\cdot}1\ \mu\mu$. On the other hand, $1\ \mu\mu = 10$ tenth-metres, &c.

TABLE OF WAVE-LENGTHS (IN TENTH-METRES).

| Substance. | Colour of line. | Wave-length. | Substance. | Colour of line. | Wave-length. |
|---|---|---|---|---|---|
| Sodium $D_1$ | Yellow. | 5896 | Strontium. | Blue. | 4607 |
| Sodium $D_2$ | ,, | 5890 | Calcium | ,, | 4226 |
| Thallium | Green | 5351 | Lithium | Red | 6708 |
| Potassium | Red | 7699 | Lithium | Orange | 6104 |
| Potassium | Violet. | 4047 | | | |

**Methods of Producing Spectra.**—One method of obtaining the flame spectrum of a metal has already been described. In certain cases an oxy-hydrogen jet is substituted for a Bunsen flame. It is possible, however, to obtain many more lines, and much greater brightness in the resulting spectrum, if the metal or its salt is introduced into an electric arc. When the arc is formed between carbons, the resulting spectrum will contain lines and flutings due to the carbon and its compounds, as well as some due to unavoidable impurities (such as sodium and calcium) and others due to the constituents of the atmosphere. A purer spectrum may be obtained by forming the arc between rods of the metal to be examined. Spectra are often obtained by producing sparks, by means of an induction coil, between pointed rods of the metal to be examined. Particles of the metal are torn off during the passage of the spark, and so give a characteristic coloration to the latter. It is sometimes convenient to produce sparks between a piece of platinum wire and the surface of an aqueous solution of a metallic salt. The latter may be contained in a small test-tube (A, Fig. 181), into the lower extremity of which a platinum wire, *f*, is sealed. This wire is connected with the negative terminal of the induction coil. An insulated platinum wire, *B*, is adjusted so that its free end, *d*, is just above the liquid; this wire is connected with the positive terminal of the coil.

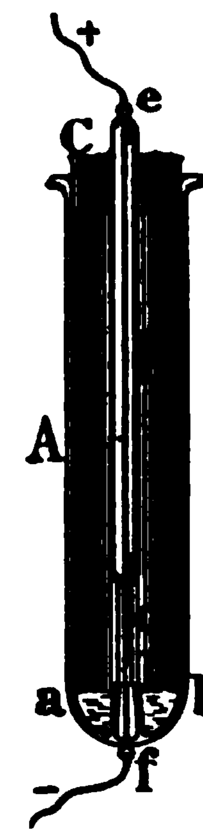


FIG. 181.—Arrangement for producing Sparks between a Wire and a Liquid.

One of the best methods of producing line spectra is by the aid of a vacuum tube. Platinum terminals are generally used, and the electric discharge from an induction coil is sent between these through a rarefied gas or vapour. The vacuum tube may take the form of a capillary tube joining two bulbs into which platinum terminals are sealed (Fig. 182). The discharge passes through the capillary tube, and the gas there becomes brightly luminous. This method is particularly suitable for determining the spectra of gases, such as oxygen, hydrogen, nitrogen, argon, &c. It can also be used in the case of the more volatile metals, such as cadmium, mercury, &c.

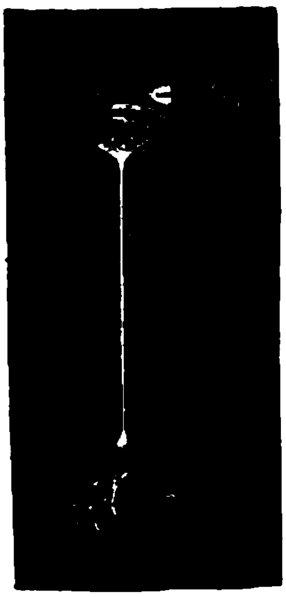

FIG. 182.—Vacuum Tube.

**Monochromatic Illumination.**—The readiest method of obtaining light approximately monochromatic is to introduce common salt into a Bunsen flame. An iron wire ring of about $\frac{3}{4}$ inch diameter is overwound with asbestos cord, and then covered with a paste made from common salt moistened with water. If the Bunsen flame is allowed to pass through this ring, a bright and constant yellow flame is obtained.

The sodium flame is only approximately monochromatic, since the spectrum comprises two adjacent lines about equal in intensity. When a strong illumination by monochromatic light is required, a mercury lamp is now generally used. Fig. 183 represents a mercury lamp due to Arons and Lummer.[1] The tube *ab* is of frosted glass, except over the ends *a* and *b*, which are clear. The side tubes *d* and *c* contain mercury, and platinum wires *m*, *p*, effect electrical connections with the mercury

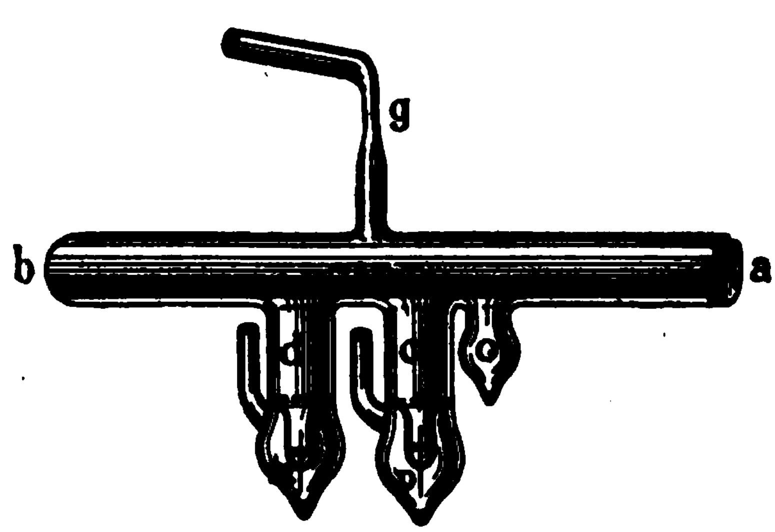


Fig. 183.—Mercury Lamp.

1 "Mercury Vacuum Lamps for Spectroscopic Work," O. Lummer, *Zeitschr. Instrumentenk.* 21, pp. 201–204, July, 1901.

cups surrounding them. *o* is a mercury reservoir, used in adjusting the quantities of mercury in *d* and *c*. The tube *ab* is thoroughly exhausted and sealed at *g*. When the terminals *m* and *p* are connected to a source of electrical supply, an arc can be struck between the mercury electrodes *d* and *c* by tilting the tube. In order to prevent overheating, the whole of the tube, with the exception of the ends *a* and *b*, is immersed in flowing water. Prof. Lummer works with an arc 3 cms. long, and uses a series resistance of 5 ohms in a 110 volt circuit, the current amounting to 16 ampères. The light is observed through the clear ends of the tube. The light from a mercury lamp when examined spectroscopically, consists of two faint lines, 5790 and 5770, in the yellow, a very brilliant green line, 5461, and a faint line, 4358, in the violet. The resulting illumination is almost entirely due to the green line, and is the nearest approach to monochromatic light that can readily be obtained.

**Characteristics of Emission Spectra.**—The spectrum emitted by a glowing vapour depends on the nature of the vapour, and also on the conditions under which emission occurs. Every element appears to have a characteristic spectrum, by observing which the presence of the element in a mixture can be inferred with certainty. Nevertheless, the same substance can, under different conditions, give rise to spectra which are entirely different. There are three characteristic classes of spectra.

**A Continuous Spectrum** presents the appearance of an unbroken luminous band, varying in colour from point to point, and shading off on both sides of a certain point at which the intensity is a maximum. The point of maximum intensity is shifted toward the violet end of the spectrum as the temperature of the radiating substance is raised.

**A Fluted Spectrum** consists of a number of broad luminous bands, sharply defined at one edge, and shading off gradually at the other edge (Fig. 184). When examined by a spectrometer of great dispersive power, each fluting is found to consist of a considerable number of lines, closely packed toward the definite edge of the fluting, and more and more widely spaced as the blurred edge of the fluting is approached.

**A Line Spectrum** consists of a number of sharply defined lines (Fig. 185) which may possess a certain obvious regularity of arrangement, or may be scattered, seemingly without any order,

over the range of the spectrum. When the lines are regularly arranged, they are said to form a **series**. Balmer has shown that the wave-lengths corresponding to seven lines in the spectrum of hydrogen can be found by substituting the values 3, 4, 5, . . . . 10, 11, for $m$ in the general formula—

$$\lambda = 3647 \frac{m^2}{m^2 - 4}.$$

Thus, substituting $m = 3$, we obtain—

$$\lambda = 3647 \frac{9}{9-4} = \frac{9}{5} \times 3647 = 6564,$$

which is the wave-length, in tenth-metres, corresponding to the red C line of hydrogen.

Substituting $m = 4$, we obtain—

$$\lambda = 3647 \frac{16}{16-4} = \frac{4}{3} \times 3647 = 4862,$$

which is the wave-length corresponding to the blue F line of hydrogen.

Substituting $m = 6$, we obtain—

$$\lambda = 3647 \times \frac{36}{36-4} = \frac{9}{8} \times 3647 = 4102,$$

which is the wave-length corresponding to the violet $h$ line of hydrogen.

To test Balmer's formula, the spectrum of hydrogen has been carefully re-examined, with the result that several lines, previously unobserved on account of their faintness, have been found in the positions indicated by the formula.

The spectra of a large number of elements, such as Na, Li, K, Rb, Cs, Ag, Mg, Ca, &c., have been found to be capable of expression by a formula essentially similar to that of Balmer. In many cases a line spectrum of an element comprises two or more series of a character similar to the above.

**Conditions for Production of Spectra of different kinds.**—If we now inquire as to the conditions under which fluted and line spectra are respectively produced, it may be stated that, as a general rule, chemical compounds, such as cyanogen, give fluted spectra, while simple substances give line spectra. This has led to the supposition that line spectra are due to elements in the atomic state, while fluted spectra are due to elements, or their compounds, in the molecular state. On the other

hand, mercury vapour, which is known to be monatomic, can be caused to give a fluted spectrum. It is even possible to cause a substance to emit both spectra simultaneously. Monkhœven used a vacuum tube containing rarefied nitrogen, and sent two separate discharges through the capillary tube; one discharge was obtained by using an ordinary induction coil, and the other by using a more powerful coil in the circuit of which a Leyden jar and a wide spark gap were included. Two spectra, one of which was fluted, while the other consisted of lines, were seen simultaneously. The line spectrum was due to the discharge from the circuit containing the Leyden jar. Generally speaking, a substance which in a vacuum tube can give either a line or a fluted spectrum, will give the line spectrum when the electrical discharge is most violent; the opposite, is, however, the case with mercury. It has, from

FIG. 184.—Spark Spectra of Nitrogen. Bright line spectrum, produced by high tension discharge, above; fluted spectrum, produced by low tension discharge, below. (From a photograph by Mr. C. P. Butler.)

this circumstance, sometimes been conjectured that a substance which gives a fluted spectrum at a low temperature, will give a line spectrum at a high temperature. Monkhœven's experiment apparently negatives this supposition. Michelson has proved that when hydrogen is raised to a temperature of 300° C., the lines of the spectrum are considerably broadened. This shows that in ordinary circumstances the gas emitting radiations in a vacuum tube is not at a high temperature. On the whole, it appears that the emission of radiations by a gas in a vacuum tube is not dependent on the temperature, but on the disruptive action of the electrical discharge.

Pringsheim has investigated the conditions under which a sodium compound emits waves corresponding to the D lines. He found that, if sodium carbonate is heated in a vessel containing only a neutral gas, such as nitrogen, no visible radiations are emitted, even at the highest temperature which he

could employ (that of the fusion of nickel). On introducing hydrogen, however, the ordinary yellow light was at once emitted. This experiment points to the conclusion that line spectra (and possibly fluted spectra also) are only produced when chemical changes are occurring in the radiating substance.

**Physical Cause of Radiations.**—As to the physical agency which produces the ethereal waves corresponding to the emitted light, certain definite conclusions have been arrived at. In the first place, the ether is probably disturbed by the motions of indefinitely small material particles which vibrate in definite periods. Recent experiments indicate that, at any rate in some cases, it is only a small fraction of an atom which vibrates, and so produces periodic disturbances in the ether. In the case of a gas or vapour emitting radiations, the **electron** (as the vibrating particle is termed), when set in motion, continues for a considerable time to vibrate regularly in its natural period, just as a bell does after being struck. During this time ether waves radiate from the neighbourhood of the vibrating electron, just as sound-waves radiate from a vibrating bell. In some cases an atom may possess a considerable number of electrons. It has been estimated that the electron which, by its vibration, produces the radiations from incandescent sodium vapour, comprises only a one-five-hundreth part of the sodium atom.

As a general rule, a heated liquid (such as melted platinum) or a solid, emits light which gives rise to a continuous spectrum. In some cases the nature of the radiations emitted depends only on the temperature of the substance; this is the case with carbon, platinum, &c. A Welsbach (or Auer) mantle, when heated, emits much brighter light than a solid (such as platinum) would emit at an equal temperature. Here there is evidence that a chemical change occurs, for the characteristic bright light is not emitted from a Welsbach mantle if this is heated in a vacuum or when surrounded by nitrogen. When a solid is heated, it appears probable that the electrons are incapable of vibrating freely. Owing to collisions or some similar cause, they are violently thrown from side to side, and produce arbitrary disturbances in the ether. The resultant disturbance, due to the irregular motions of numberless electrons, travels through the ether and is

analysed into its harmonic constituents by a prism. Theory shows that the wave-length corresponding to the harmonic constituent of greatest amplitude will be smaller in proportion as the arbitrary disturbances are more violent and more frequent. If we remember that the velocity with which the molecules of a body are moving increases with the temperature, we can, in a general way, see why it is that the point of maximum intensity in a continuous spectrum is shifted toward the violet as the temperature of the radiating body is raised.

**Absorption.**—When light is incident on the surface of a transparent medium, part of the light is reflected, and the rest is transmitted unchanged. Certain material media, however, act in a very different manner toward light. When light is incident on lamp-black or platinum-black, it is neither reflected nor transmitted; the light is absorbed, and ceases to exist as light. Lamp and platinum black absorb waves of all lengths, except the very longest. This kind of absorption is termed **general.**

Certain substances strongly absorb light corresponding to a particular part of the spectrum, and transmit the remaining light unchanged. Such absorption is termed **selective.** It produces black bands in the spectrum of the transmitted light; these are termed **absorption bands.** A piece of ruby glass (which is coloured with oxide of copper) transmits the red rays, and absorbs the light corresponding to the remainder of the spectrum.

EXPT. 56.—In front of the slit of a spectroscope place a test-tube containing an aqueous solution of gamboge yellow. The resulting spectrum is seen to comprise only yellow and green light. In a similar manner observe that an aqueous solution of Prussian blue transmits only green and blue light. On mixing the above coloured solutions, a green liquid is obtained, *i.e.*, one which transmits only green light.

This experiment explains how it is that a mixture of blue and yellow pigments forms a green pigment. The blue pigment absorbs all rays except the green and blue, and the yellow pigment absorbs all rays except the yellow and green. The mixture absorbs all rays except the green, which are transmitted by both pigments. In the case of water-colour painting, the light reaching the white paper through a layer of

paint made from a mixture of yellow and blue, is almost entirely composed of green rays, and these, being reflected from the paper, give the latter a green coloration.

The aniline dyes generally exhibit strong selective absorption. Fuchsine (generally known to microscopists as magenta) strongly absorbs yellow and green light, the transmitted light, composed of blue and red, being of a rich purple colour.

Kirchhoff formulated the following important law: **A substance which emits waves of definite periods when heated, will selectively absorb waves of the same periods when cool.**

This law was arrived at by a study of the absorption of sodium vapour. As already stated, incandescent sodium vapour emits light corresponding to two adjacent lines (the D lines) in the yellow part of the spectrum. According to Kirchhoff's law, sodium vapour should also absorb light corresponding to these two lines. That it does so may be proved by the following experiment.

EXPT. 57.—Illumine the slit of a spectrometer with lime light. A bright continuous spectrum is thus formed. Now place the flame of a spirit-lamp (the wick of which has previously been soaked in salt solution and then dried) immediately in front of the slit, so that the white light has to traverse the spirit-lamp flame before reaching the slit. Two narrow black lines, exactly coinciding with the position of the D lines, will be seen in the spectrum.

For this experiment to succeed, the temperature of the absorbing vapour must be lower than that of the source of white light, as otherwise the sodium vapour will emit more light than it absorbs, by the Second Law of Thermodynamics.[1] Thus, a spirit-lamp flame must be used when the illuminant is lime-light. If an arc-lamp is used as a source of white light, then a Bunsen flame in which a small quantity of sodium is burnt may be used as an absorbent. Similar effects may be produced by placing a small piece of sodium in the arc itself; in this case it is the cooler sodium vapour, surrounding the incandescent carbon, which effects absorption.

The physical explanation of Kirchhoff's law is very simple. For waves in the ether to be produced by the vibration of a

[1] *Heat for Advanced Students*, by the Author, p. 339.

sodium electron, there must be some mechanical connection between the two, so that the motion of one disturbs the other. Thus, light-waves tend to move the sodium electrons, and when the periods of the waves are equal to the free periods of the electrons, the latter will be set in violent vibration, and will thus absorb energy (p. 255). Thus, the white light is robbed of the waves agreeing in period with those of the sodium electrons.

Kirchhoff's law is of a perfectly general character, and applies to the emission and absorption of light by a substance of any kind.

**The Solar Spectrum.**—When the slit of a spectrometer is illumined by sunlight, the resulting spectrum is seen to be crossed by a considerable number of fine black lines (Fig. 185). These lines were first observed by Wollaston in 1802; they were carefully studied at a later date by Fraunhofer, and are thence termed **Fraunhofer lines.** With a given spectrometer the Fraunhofer lines occupy positions in the spectrum which are perfectly definite, and thus correspond to certain missing light-waves.

The following table gives the wave-lengths of the most prominent Fraunhofer lines; by observation of these lines a spectrometer may be calibrated in a manner similar to that described on p. 331.

WAVE-LENGTHS OF FRAUNHOFER LINES (IN TENTH-METRES).

| Line. | No. of components. | Part of spectrum. | Element to which line corresponds. | Wave-length. |
|---|---|---|---|---|
| A | 1 | Extreme red | Oxygen | 7594 |
| B | 1 | Red | ,, | 6867 |
| C | 1 | ,, | Hydrogen | 6563 |
| $D_1$ | 1 | Orange | Sodium | 5896 |
| $D_2$ | 1 | ,, | ,, | 5890 |
| E | 3 | Green | Iron and Calcium | 5270 |
| *b* | 1 | ,, | Magnesium | 5184 |
| F | 1 | Blue | Hydrogen | 4861 |
| G | 2 | ,, | Iron and Calcium | 4308 |
| *h* | 1 | Violet | Hydrogen | 4102 |
| H | 1 | ,, | Calcium | 3969 |
| K | 1 | ,, | ,, | 3934 |

The explanation of the Fraunhofer lines was first given by Kirchhoff, who observed that many of these lines coincided with bright lines in the spectra of the elements (Fig. 185). The sun is assumed to consist of an incandescent solid or liquid nucleus, surrounded by a cooler envelope in which oxygen, hydrogen, iron, calcium, &c., are present in the form of gases or vapours. The vapour of an element absorbs the waves which it would emit if it were incandescent, and thus, the white light emitted by the solar nucleus is robbed, in passing through the enveloping layer, of those waves which vibrate in the same periods as the elements there present.

Fraunhofer observed 576 lines in the solar spectrum; many more lines have since been observed. Most of these have been


FIG. 185.—Solar Spectrum, showing Fraunhofer Lines (D lines to extreme right, H and K lines to extreme left). Comparison spectra of hydrogen (above) and helium (below). (From a photograph by Mr. C. P. Butler.)

found to correspond to lines in the spectra of elements present on the earth. Consequently we have good grounds for believing that the chemical constitution of the sun is similar to that of the earth.

It will always remain a matter for some surprise that the presence of the dark lines in the solar spectrum was not observed by Newton. Very common prisms will serve to make some, at any rate, of the Fraunhofer lines visible; an ordinary prism from a candelabrum will suffice for this purpose. There appears to be no doubt that, under the conditions of his investigations, the dark lines should have been seen. He was quite aware that, if light from a small circular hole is merely

refracted by a prism and then allowed to fall on a white screen, the coloured image consists of numberless small coloured circles which overlap. To avoid this, he says: "That those circles may answer more directly to that hole, a lens is to be placed by the prism to cast the image of that hole upon the paper." He then adds: "Yet instead of the circular hole, 'tis better to substitute an oblong hole shaped like a long parallelogram, with its length parallel to the prism. For if the hole be an inch or two long, but a tenth or twentieth part of an inch broad, or narrower, the light of the image [*i.e.* the spectrum] will be as simple as before or simpler, and the image will become much broader, and therefore, more fit to have experiments tried in its light than before." A possible explanation of Newton's failure to observe the dark lines is that he used an assistant for observing the spectra in certain experiments. "An assistant, whose eyes for distinguishing colours were more critical than mine, did by right lines, . . . . drawn across the spectrum, note the confines of the colours."[1]

**Stellar Spectra.**—In examining the spectra of the stars, a considerably simplified apparatus may be used. In the first place, since the stars appear as mere points of light, at a distance from the earth which is practically infinite, the light-waves arriving at the earth will be sensibly plane, so that no slit or collimator is required. A prism is mounted in front of the object-glass of the telescope, and a narrow coloured line, the spectrum of the star, is seen through the eye-piece. When it is sought to photograph the spectrum of a star, the refracting edge of the prism is adjusted to be exactly parallel to the direction of the apparent motion of the star due to the rotation of the earth. In these circumstances the apparent motion of the star merely broadens the spectrum without destroying any of its detail. The small approximately straight line joining the initial to the final position of the star corresponds to the linear slit of a spectrometer.

The stars are found to possess definite spectra (Fig. 186), some of which, in general appearance, resemble the solar spectrum, while others are more nearly allied to the fluted spectra of some elements. Dark lines corresponding to a number of elements present on the earth (including hydrogen) have been observed. The spectra of the nebulæ consist entirely of bright lines; this

[1] "On the Formation of a Pure Spectrum by Newton," G. Griffith, *B. A. Report*, 1885, p. 940.

seems to show that the nebulæ are masses of incandescent gas, which have not, as yet, cooled so far as the sun has, so as to acquire solid (or liquid) nuclei.

**Invisible Portions of the Spectrum.**—The portion of the spectrum which can be directly perceived by the aid of our eyes is comprised between the wave-lengths 3930 (violet) and 7594 (red). It has been proved, however, that the ethereal waves

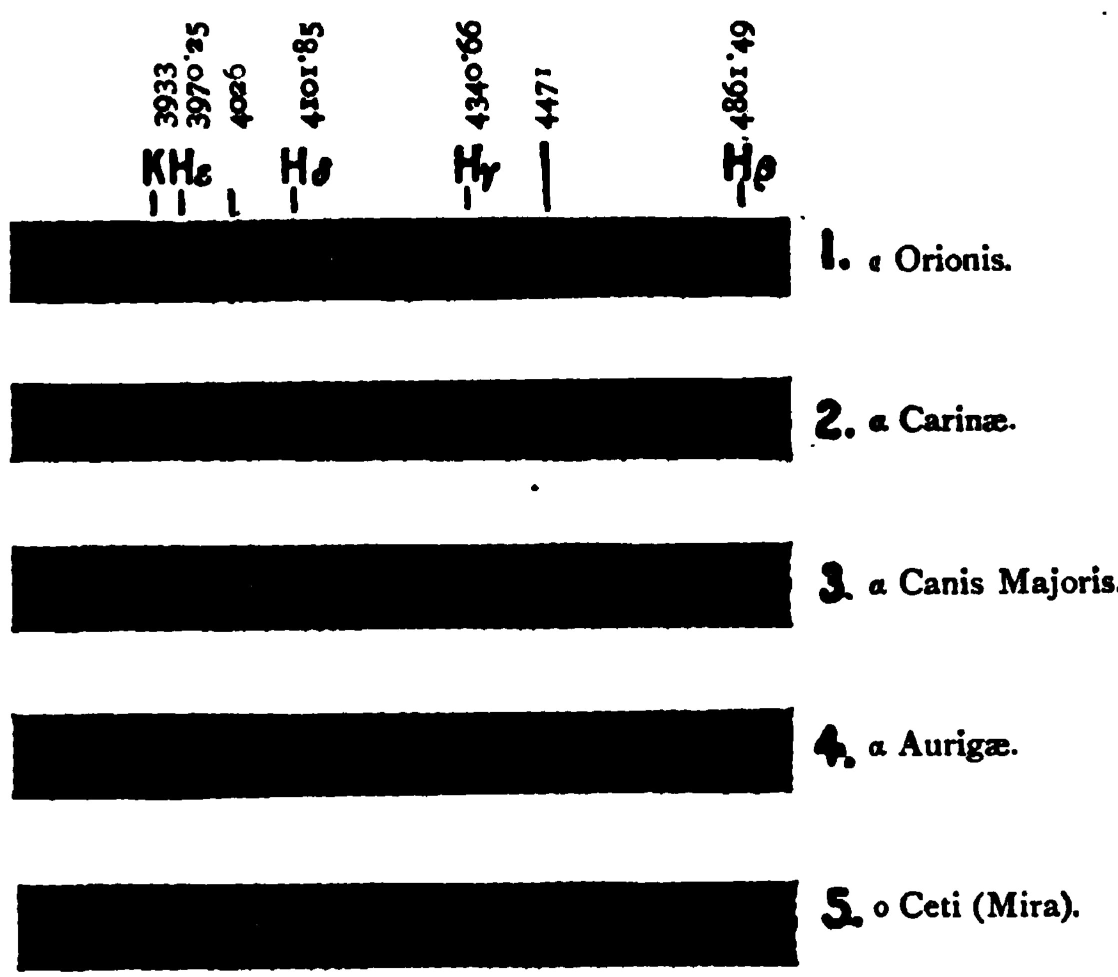


FIG. 186.—Stellar Spectra. (1) ε *Orionis*, continuous spectrum with narrow hydrogen lines; (2) α *Carinæ*, continuous spectrum with broader hydrogen lines; (3) α *Canis Majoris*, continuous spectrum with very broad (or *enhanced*) hydrogen lines; (4) α *Aurigæ*, spectrum resembling that of sun; (5) o *Ceti*, spectrum composed of bright lines and flutings. (From photographs by Mr. C. P. Butler.)

which can affect the eye, and so produce vision, form only a small portion of those reaching us from the sun. The spectrum, in fact, extends beyond its visible limits at both ends.

The portion of the spectrum extending beyond the violet, (termed the **ultra-violet spectrum**) can be directly observed by the aid of photography. The short waves corresponding to this portion of the spectrum are particularly active in decomposing salts of silver, and have, for this reason, been termed *actinic*.

The ultra-violet spectrum can be photographed without difficulty, if a prism is used which does not absorb waves of short length. A glass prism is found to be unsuitable for this purpose, since it is practically opaque to the greater portion of the ultra-violet radiation ; a quartz prism is, however, found to be transparent for short waves, and so is universally used. It has also been found that atmospheric air absorbs the ultra-violet radiations ; by experiments conducted *in vacuo*, the properties of ultra-violet waves as short as 1,000 tenth-metres have been observed.

The ultra-violet solar spectrum is found to be continuous, like the visible spectrum, and also to comprise a number of dark lines. Ultra-violet radiations have been found to be reflected and refracted according to the usual laws, and can be caused to interfere, and can be polarised by tourmaline, like ordinary light.

The existence of a portion of the spectrum extending beyond the extreme end of the visible red was first observed by Herschel in 1800 ; it is termed the **infra-red spectrum.** Herschel found that a thermometer with a blackened bulb, when placed at a point some distance beyond the red end of the visible spectrum, indicated a rise of temperature which proved that radiations were reaching it. To study the infra-red spectrum we need, in the first place, a prism of a substance which does not absorb radiations of long wave-length ; in the second place, we need an instrument which will indicate a very small rise of temperature due to the radiations absorbed. Prisms of rock-salt, sylvine (a crystalline form of potassium chloride resembling rock-salt), or fluor-spar are generally used ; quartz prisms may be used for *very long* wave-lengths. The instruments used for absorbing the radiations, and indicating the consequent rise of temperature produced, are the thermopile, the radio-micrometer, and the bolometer.[1]

The most elaborate study of the infra-red solar spectrum has been carried out, in the course of many years, by the American physicist, Prof. Langley, by the aid of his bolometer. This consists of two blackened strips of platinum, about a tenth of a millimetre in breadth, and a hundredth of a millimetre in thickness, arranged to form two arms of a Wheatstone's bridge.

[1] For a description of these instruments the student is referred to *Heat for Advanced Students*, by the Author, Chapter XIX.

The usual galvanometer and battery connections having been made, the resistances in the remaining arms are adjusted so that the galvanometer shows no deflection when the platinum strips are at the same temperature. A solar spectrum is formed, in the usual way, by the aid of a rock-salt prism, all lenses being also of rock-salt or fluor-spar. The prism is caused to rotate slowly by clockwork of the greatest precision, so that one part after another of the spectrum passes across one of the platinum strips, the other strip being screened from radiations. The exposed strip is adjusted to be parallel to the spectrometer slit and to the refracting edge of the prism; when radiations fall on it, a rise of temperature ensues, and the balance of the bridge being destroyed, a deflection of the galvanometer is produced. The sensitiveness of the arrangement is such that a rise of temperature amounting to no more than *one hundred-millionth of a Centigrade degree* produces a readable deflection. What would be a dark band in the spectrum, could our eyes be affected by the long infra-red waves, will fail to heat the platinum strip, and the galvanometer deflection will be diminished or reduced to zero. A spot of light, reflected from a mirror attached to the galvanometer needle, falls on a photographic plate which is caused to move up or down at a speed proportional to the rate of rotation of the prism; for instance, the photographic plate moves vertically through a centimetre while the prism rotates through one minute of arc. In this way, as the spectrum slowly passes across the exposed strip of platinum, the galvanometer registers its own deflection, which tells us whether a "bright" or "dark" band is focussed on the strip. Two of the resulting curves are shown in the upper part of Fig. 187. The spectrum in the lower part of the figure is what we should see if our eyes were sensitive to the long infra-red waves. The wave-lengths of a number of characteristic lines are given in microns (p. 331). It will be seen that Langley has investigated the infra-red spectrum through a range extending from $0{\cdot}76\mu$ (7,600 tenth-metres) to $5{\cdot}3\mu$ (53,000 tenth-metres). Thus, while the eye is only sensitive to light-waves comprised in a little less than a single octave, the bolometer has made us acquainted with an additional three octaves of the solar spectrum.

Our knowledge of the infra-red solar spectrum is not wholly derived from the work of Langley. In 1880, Sir William (then

Captain) Abney obtained a photograph of the infra-red solar spectrum extending to a wave-length equal to $1{\cdot}1\mu$ (the absorption band marked $\phi$, Fig. 187). Photography of the infra-red spectrum has never been carried beyond this point. Previous to 1881 it was generally believed that the infra-red solar spectrum ended at $1{\cdot}8\mu$ (the absorption band marked $\Omega$, Fig. 187); at that date Langley discovered the extension of the spectrum to $5{\cdot}3\mu$.

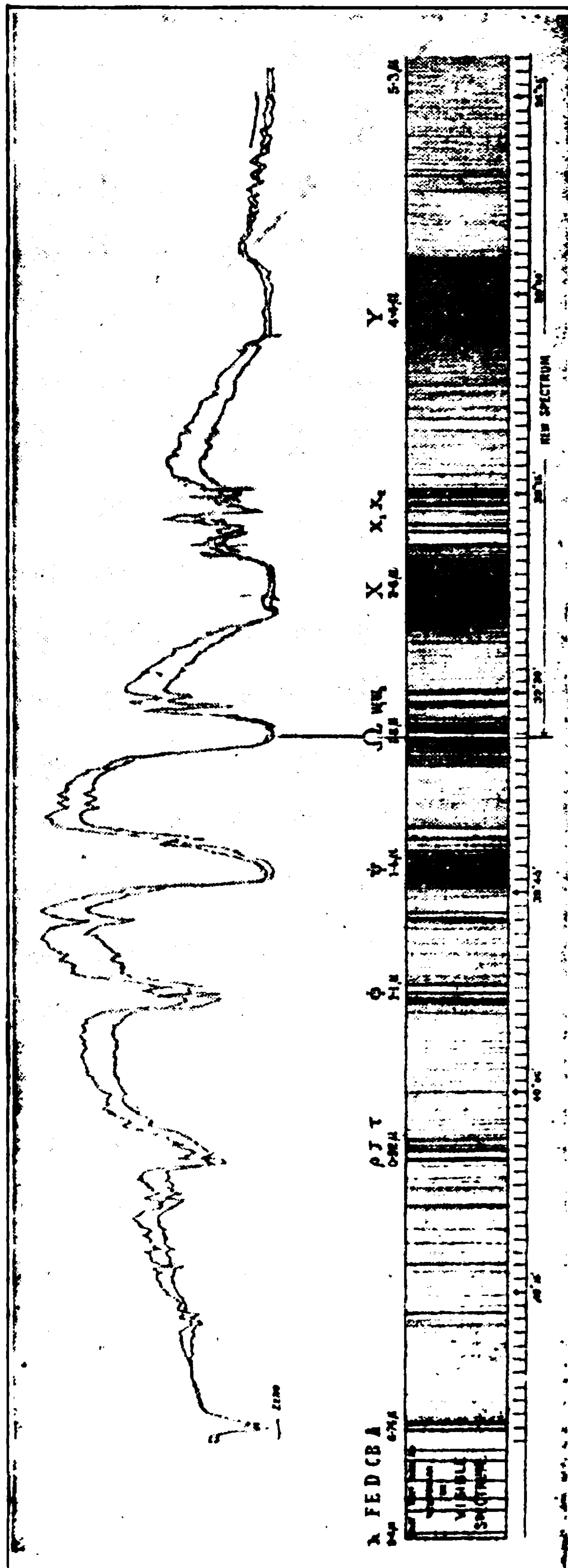

Fig. 187.—Langley's Bolometer Records, and Infra-Red Solar Spectrum.

**Present Knowledge of Infra-Red Radiations.**—During recent years several most interesting investigations have been carried out in connection with the infra-red radiations derived from terrestrial sources; some of these investigations will be described subse-

quently. Fig. 188 represents, in two different ways, the range within which the properties of ether waves have been investigated. In the upper diagram the horizontal scale is graduated in microns, and shows the limits of infra-red radiations which have been studied up to the present. The narrow white line, V, to the left represents the visible spectrum; the adjoining shaded portion represents the infra-red solar spectrum studied by Langley. Recent experiments with infra-red waves radiated by heated bodies have extended our knowledge to include wave-lengths up to $61.3\mu$ (S, Fig. 188). The difficulties of experimenting with these very long waves is considerable. Fluor-spar becomes practically opaque to radiations at $f$ (Fig. 188), or $11\mu$; rock-salt and sylvine are opaque to radiations beyond $r$ and $s$ respectively. Quartz, which possesses strong absorption bands for $\lambda = 8\mu$ and $\lambda = 21\mu$, and is thus opaque to comparatively short wave-lengths, is transparent for wave-lengths in the neighbourhood of $50\mu$. The lines R and S represent wave-lengths which are very strongly reflected from rock-salt and sylvine respectively. $F_1$ and $F_2$ represent wave-lengths which are strongly reflected from fluor-spar.

Rubens, Nichols, and Aschkinass, to whom we are indebted for our knowledge of the properties of infra-red waves of very great length, find that paraffin, benzine, and carbon bisulphide are transparent throughout the whole range of the infra-red spectrum; on the other hand, water is opaque through the same range. Paschen has found that the carbon-dioxide and aqueous vapour in the atmosphere exercise marked selective absorption on waves of lengths $2.8\mu$, $4.3\mu$, and $5.9\mu$; the wholesale absorption of infra-red waves by aqueous vapour, which Tyndall claimed to have discovered, has not been confirmed.

In the lower diagram (Fig. 188) an arrangement different from that in the upper one is used. Instead of employing a scale proportional to wave-lengths, the scale is graduated in octaves. Thus the white portion represents the visible spectrum (about 1 octave, from $\lambda = 0.4\mu$ to $\lambda = 0.8\mu$). To the left are the two octaves of ultra-violet waves which have been investigated. To the right of the visible spectrum are about $6\frac{1}{2}$ octaves, between which limits the infra-red radiations have been studied.

It has been found that when an electric spark passes between two small metal balls, ether waves of considerable length are emitted; it is these waves which are utilised in wireless telegraphy. The shortest waves of this kind which have been obtained possess a wave-length of about 3 mm. In other respects than wave-length, electrical waves appear to be exactly similar to the waves producing the ultra-violet, visible, and infra-red spectra. It will be seen that the gap between the extreme infra-red rays and the shortest electrical waves has been reduced to very reasonable dimensions. It is interesting to note that Rubens found the properties of the infra-red waves for $\lambda = 50\mu$ to agree much better with those of electrical

FIG. 188.—Range through which Ether Waves have been investigated.

waves than with those corresponding to the visible spectrum. Thus, substances like water, which are opaque to electrical waves, are also opaque to the long infra-red waves. Benzine, paraffin, carbon-bisulphide, and gutta-percha are transparent to both the long infra-red and the electrical waves. Using an acute-angled prism, Rubens determined the refractive index of quartz for waves of length, $\lambda = 56\mu$. He found this to be equal to 2·18. For electrical waves the refractive index is equal to 2·12, while for visible light the refractive index is equal to 1·5.

**Distribution of Energy in the Spectrum.**—When waves are absorbed by a blackened surface, their energy is converted into molecular energy, or heat. This is the principle underlying almost all investigations dealing with the infra-red spectrum. Thus, the deflection of the galvanometer used in conjunction with a bolometer, is proportional to the rate at which energy is being communicated to the strip. It is found that at all points in the spectrum the above energy transformation occurs; in other words, all waves are vehicles of energy. The distribution of energy in the solar spectrum has been determined. From Langley's curve (Fig. 187) it would at first sight appear that the solar radiations which possess the greatest energy were infra-red waves. It must be noticed, however, that the horizontal scale below Fig. 187 is not graduated in wave-lengths, but in degrees and minutes of rotation of the prism; thus, the distance between two dark lines is not proportional to their difference of wave-length. Toward the right the spectrum is compressed, while toward the left it is drawn out; consequently a greater number of waves of different lengths must have simultaneously fallen on the bolometer strip in the extreme infra-red than in the portion nearer to the visible spectrum. Hence, the point of maximum energy has been shifted toward the infra-red. When the energy curve for solar radiations is drawn with wave-lengths as abscissæ (instead of deviations, as in Fig. 187), the point of maximum energy is found to lie within the visible spectrum between the F and D lines. The general form of the curve is shown by the irregular line in Fig. 189; the distance between V and R represents the visible spectrum. The dotted line represents the probable distribution of energy in the solar radiations before these have been robbed by the selective absorption of the sun's cool enveloping layer, and of the earth's

atmosphere. The remaining curves show the distribution of energy in the spectrum of a luminous gas flame, and in that of the incandescent positive crater of an electric arc-lamp.

FIG. 189.—Distribution of Energy in the Spectra of Sunl[illegible] Electric Arc light, and Gas light.

Lord Kelvin estimates that every square foot of th radiates energy at the rate of 7,000 horse power. experimenting under the cloudless sky of Egypt, found tl metre of the earth's surface receives energy at the rat power. The whole earth receives energy from the st 130 billion (*i.e.* $130 \times 10^{12}$) horse power. Assuming tl the earth to be 1,500 millions, the power per human from the sun amounts to 80,000 horse-power!

**The Doppler Effect.**—When a locomotive engine, sounding its whistle, runs at high speed through a railway station, an observer on the platform may note an abrupt fall in the pitch of the whistle as the engine passes him. The change in the pitch depends on the speed of the engine, being enhanced by an increase of speed. Since the pitch of a note depends on the period of the sound-waves, and the period, T, of the waves is connected with the wave-length, $\lambda$, by the equation (p. 261)—

$$VT = \lambda,$$

it follows that the length of the sound-waves reaching the observer is shorter when the engine is approaching than when it is receding. A similar effect may be noted by an observer on a train as he passes a stationary engine which is blowing its whistle. The pitch of the whistle is higher while he is approaching than when he is receding from the stationary engine. Thus this phenomenon is obviously due to a *relative*

*motion* between the source of the sound and the observer. The law deduced from these phenomena is as follows:—**Relative motion between an observer and a source of wave disturbance increases or decreases the apparent period of the emitted waves, according as the motion increases or decreases the distance between source and observer.** The magnitude of the apparent change depends on the relative velocity between the two.

This result is in perfect accord with the wave theory. Let us first examine the effect produced by the motion of a wave source. Let the upper half of Fig. 190 represent the simultaneous positions of three spherical wave crests radiating from the stationary point O; a fourth crest is supposed to be just on the point of formation at O. The circles $W_1$, $W_2$, and $W_3$ are concentric; thus, the wave-length will be equal to the distance between two neighbouring circles, and is the same in all directions.

FIG. 190.—Döppler Effect, due to Motion of the Wave Source

The lower half of Fig. 190 represents three spherical wave crests generated by a source of harmonic disturbance travelling along the line OC with a uniform velocity. The wave $w_1$ originated from a disturbance at O. The velocity of a wave depends only on the nature of the medium through which it is transmitted; thus, since the medium is supposed not to be in bodily motion, the wave $w_1$, at the end of a short interval of time, occupies the same position as if the source of disturbance were stationary at O; consequently, $w_1$ and $W_1$ are parts of the same sphere. When the wave $w_2$ originated, the source of disturbance was at A. Hence $w_2$ is a sphere of radius equal to that of $W_2$, but with centre at A. The wave $w_3$ originated from a disturbance at B, where AB = OA = the distance travelled over by the source during a time equal to the period of the harmonic disturbance. Hence $w_3$ is a sphere of radius equal to that of $W_3$, but with centre at B. A fourth wave is just

on the point of starting from C, where BC = AB. Directly ahead of the moving source, the distance between two crests, or the wave-length, is less than the length of the waves emitted by the stationary source. In the rear of the moving source the wave-length is greater than the length of the waves emitted by the stationary source. Hence at a point *toward* which the source is travelling the wave-length will be smaller than if the source were stationary; at a point *from* which the source is travelling, the wave-length will be greater than if the source were stationary. Since the velocity of wave transmission is constant for all directions, it follows that the period of the waves is proportional to the wave-length; hence we obtain the law already enunciated.

Let V be the velocity of wave transmission, while $v$ is the velocity of the source. If $T_0$ is the period of the harmonic disturbance, then $OA = vT_0$. Let $\lambda_0$ be the length of the waves emitted by the stationary source, while $\lambda_1$ is the length of the waves directly ahead of the moving source. Then $\lambda_1 = \lambda_0 - OA = VT_0 - vT_0 = (V - v)T_0$.

$$\text{But } T_0 = \frac{\lambda_0}{V}. \quad \text{Thus, } \lambda_1 = \frac{V - v}{V}\lambda_0.$$

If $T_1$ is the period of the waves arriving at a point directly ahead of the moving source, $\lambda_1 = VT_1$. Then—

$$\lambda_1 = VT_1 = (V - v)T_0, \text{ and } T_1 = \frac{V - v}{V}T_0.$$

If $\lambda_2$ and $T_2$ respectively denote the wave-length and period of waves in the rear of the moving source, it can be proved in a similar manner that—

$$\lambda_2 = \lambda_0 + OA = (V + v)T_0 = \frac{V + v}{V}\lambda_0; \text{ while } T_2 = \frac{V + v}{V}T_0.$$

Let us now examine the effect of the motion of the observer on the apparent wave-length and period of the incident waves. Let the observer be travelling with a uniform velocity $v$ from left to right (Fig. 191), while plane waves $W_1$, $W_2$, $W_3$ . . . are travelling from right to left with a velocity V. Let the observer meet the wave $W_1$ at O; then, as he travels forward toward the right, he will meet $W_2$ at some point A, and $W_3$ at B, where OA = AB. Let $T_0$ be the true period of the waves; then, if the observer were stationary at O, the

wave $W_2$ would reach him $T_0$ seconds after the wave $W_1$. As it is, he travels through a distance OA before meeting $W_2$. The true wave-length $\lambda_0$ is equal to Ow. The distance OA is equal to $vT_1$, where $T_1$ is the apparent period of the waves. Also $Aw = VT_1$. Then, since $OA + Aw = Ow$, we have—

$$(v + V)T_1 = VT_0, \text{ or } T_1 = \frac{V}{V + v} T_0.$$

FIG. 191.—Döppler Effect, due to Motion of the Observer.

Thus the period of the waves is apparently diminished by the motion of the observer in a direction opposite to that of the waves.

If the observer is moving with a uniform velocity, $v$, in the same direction as the waves, it is easily proved that the apparent period, $T_2$, of the waves will be given by—

$$T_2 = \frac{V}{V - v} T_0,$$

which denotes an apparent increase in the period of the waves.

It can now readily be proved that, if source and observer are moving with equal velocities in the same direction, there will be no apparent alteration of period. Let the observer be moving along OC (Fig. 190). The period of the waves along that line is equal to $\frac{V - v}{V} T_0$, and since the observer is moving in the same direction as the waves, the observed period will be equal to $\frac{V}{V - v} \cdot \frac{V - v}{V} T_0 = T_0$, which is the period of the source of harmonic disturbance.

Finally, we must determine the alteration in wave-length produced by reflection from a moving mirror; the result will also apply to the diffusive reflection of light from a white object.

The upper half of Fig. 192 represents a train of plane waves travelling from right to left, incident on a reflecting surface, S, travelling from left to right. The lower half of the figure represents the reflected wave train. The wave $W_1$ is just in contact with the surface S,

FIG. 192.—Döppler Effect, due to Reflection from a Moving Mirror.

and the reflected wave is in the act of being formed. The length, $\lambda_0$, of the incident waves is equal to OC. Let the reflecting surface meet the wave $W_2$ at A, at a time $T_1$ seconds after meeting $W_1$ at O. At this instant a second reflected wave is formed. In the time $T_1$ the wave $W_2$ has travelled over the distance CA, so that the reflected wave due to $W_1$, travelling with an equal velocity, has reached B, where OB = CA. Thus, $BC = OA = vT_1$, where $v$ is the velocity of the reflecting surface.

If $T_0$ is the true period of the incident waves, $OC = \lambda_0 = VT_0$. The length, $\lambda_1$, of the reflected waves is obviously equal to AB.

Then—

$$VT_0 = OA + AB + BC = 2vT_1 + \lambda_1$$

Further—

$$OA + AC = vT_1 + VT_1 = VT_0. \quad \therefore T_1 = \frac{V}{V+v}T_0.$$

Then—

$$VT_0 = \frac{2vV}{V+v}T_0 + \lambda_1,$$

and—

$$\lambda_1 = VT_0\left(1 - \frac{2v}{V+v}\right) = VT_0 \cdot \frac{V-v}{V+v} = \lambda_0\frac{V-v}{V+v}.$$

Thus, since an increase in $v$ diminishes the numerator, and at the same time increases the denominator of the fraction by which $\lambda_0$ is multiplied, it follows that the diminution of wave-length produced by reflection from a moving surface is greater than that produced by motion of the wave source or of the observer. The period of the reflected waves will obviously be equal to $T_0\frac{V-v}{V+v}$.

If we suppose the reflecting surface to be travelling in the same direction as the waves, the length, $\lambda_2$, of the reflected waves will be equal to $\lambda_0\frac{V+v}{V-v}$; this may be found by reversing the sign of $v$.

The Doppler principle may be illustrated by the following experiment :—

EXPT. 58.—A tuning-fork mounted on a resonance box is required for this experiment. Bow the fork strongly, and then move it to and from a wall. When moving away from the observer, and toward the wall, the waves reaching the observer directly from the fork are increased in length, while those reaching the wall and thence reflected to the observer are decreased in length. The two wave trains produce "beats" which can be distinctly heard. If the vibrating fork is held

stationary at a short distance from a door which is moved backwards and forwards, beats will also be produced; these illustrate the alteration in wave-length produced by reflection from a moving body.

The Döppler principle can be used to explain a great number of phenomena connected with radiation. Some applications will be given in the following paragraphs.

**Stellar Motion in the Line of Sight.**—If a star is moving in a direction at right angles to an imaginary line joining the star to the earth (the line of sight), the star's position in the heavens will slowly change. The motion of a star *in* the line of sight will produce no change in its apparent position. On the other hand, since a star is a source of wave disturbance, its motion in the line of sight will modify the length of the emitted waves. As already pointed out, certain lines in stellar spectra obviously correspond to elements present on the earth. On the other hand, the wave-length of radiations emitted by a star approaching the earth should be smaller than if the star were stationary; the corresponding spectral line should be displaced toward the violet. Similarly, if the star is receding from the earth, the spectral lines should be displaced toward the red end of the spectrum.

In 1868, Dr. Huggins observed that in the spectrum of Sirius the F line (hydrogen) is slightly shifted toward the red end of the spectrum. This denotes a recession of Sirius from the earth. On measuring the exact shift of the F line, its change of wave-length became known, and applying the equation already deduced, it was found that the relative velocity between the earth and Sirius is equal to about 29 miles per second.

Further observations by Dr. Huggins showed that, while some stars are moving away from the earth, others are moving toward the earth. The stars Sirius, Rigel, Castor, Regulus, and $\delta$ Ursæ Majoris (which are situated in that part of the heavens which is opposite to the constellation of Hercules), are moving away from the earth. The stars Arcturus, Vega, and $\alpha$ Cygni (which are situated in the neighbourhood of the constellation of Hercules) were found to be moving toward the earth.

**Nature of Saturn's Rings.**—The planet Saturn is encircled by three concentric rings (Fig. 193). The middle ring is

separated from the outer one by a dark space, *Cassini's division*, through which stars are sometimes seen. The innermost ring is only faintly visible, and is termed the *crape ring*. The thickness of the rings does not exceed 100 miles, while the outer diameter of the exterior ring is about 172,310 miles, the inner diameter of the crape ring being about 88,190 miles.

The nature of Saturn's rings has been the subject of much speculation. At first sight, the most probable conjecture seems to be that the rings are solid and continuous; but Maxwell proved that in this case the rings would be unstable—that is, a small perturbing force would cause a displacement of the rings which would increase at a greater and greater rate, and the rings would finally break up into fragments.

FIG. 193.—The Planet Saturn.

According to a second theory, each ring consists of a swarm of small satellites, so closely packed that they appear to be continuous. Such an arrangement would be dynamically possible.

These two theories lead to different results as to the relative velocities of rotation of the exterior and interior edges of a ring. If a ring is solid, its outer and inner radii being respectively equal to $r_1$ and $r_2$, then the velocity of the outer edge, corresponding to a rotation of $n$ turns per second ($n$ being probably a fraction), will be equal to $2\pi r_1 \times n$. The velocity of the inner edge of the ring would similarly be equal to $2\pi r_2 \times n$. *Thus, the outer will be moving more quickly than the inner edge*, since $n$ is constant for both, while $r_1$ is greater than $r_2$.

If the rings consist of swarms of small satellites, each satellite must be in equilibrium under the action of its centrifugal force and the attraction exerted upon it by the planet Saturn. By Newton's law of gravitation, the attraction between two bodies, of masses M and $m$, is proportional to the product M$m$, divided by the square of the distance between the centres of the bodies. If M denotes the mass of Saturn, and $m$ that of a satellite at a distance $r$ from the centre of Saturn, the gravitative force acting on the satellite will be equal to $(G \cdot Mm)/r^2$, where G is the attraction between two bodies, each of unit mass, at

unit distance apart. Further, if $v$ is the velocity of the satellite, its centrifugal force is equal to $mv^2/r$. Thus, for the circular motion of the satellite to be permanent—

$$\frac{mv^2}{r} = \frac{GMm}{r^2}. \quad \therefore v^2 = \frac{GM}{r}.$$

Thus, *the satellites farthest away from the planet, forming the outer edge of a ring, will possess a smaller velocity than those nearer to the planet, forming the inner edge of a ring.*

The rings of Saturn are known not to be self-luminous, but to owe their visibility to reflected sunlight. Although this light is relatively feeble, spectroscopic analysis of it reveals some of the more prominent Fraunhofer lines. If the rings were stationary, these lines would occupy the same position as in the solar spectrum. If the rings are revolving, then the portions to the right and left of Fig. 193 will be moving in the line of sight, and Doppler effects will be produced. The spectrum of the light from that side of a ring which is moving toward the observer will show a displacement of the Fraunhofer lines in the direction of the violet end of the spectrum. This displacement, according to the satellite theory, will be greater for the inner than for the outer edge of a ring, since, according to that theory, the inner edge is moving more quickly than the outer edge. The opposite would be the case if the rings were solid and continuous.

Professor Keeler investigated this point, and obtained decisive evidence in favour of the satellite theory. The Doppler displacement was found to be greater for light from the inner than for that from the outer edge of a ring. Deslandres confirmed Keeler's results, and in addition was able to determine the velocity of rotation at the outer and inner edges of the ring system. He was also able to determine the rotational velocity of the planet itself.

Deslandres found that the rotational velocity at the equator of Saturn is 9·38 kilometres per second; the calculated value is 10·3 kilometres per second. The inner ring has a velocity of 20·1 kilometres per second (calculated 21·0); the outer ring has a velocity of 15·4 kilometres per second (calculated 17·14).

**Broadening of Spectral Lines.**—According to the kinetic theory of gases, the molecules of a gas are moving hither and

thither with great velocities. At 0° C. a hydrogen molecule on an average possesses a velocity of the order of $1 \cdot 84 \times 10^5$ cm. per second.[1] Its velocity is proportional to the square root of the absolute temperature.

When a gas or vapour is rendered luminous, it appears that a small portion of the molecule is set in periodic motion, and so disturbs the ether and produces the waves which constitute light. Thus, each molecule of a gas may be considered as possessing a source of harmonic disturbance which is carried with it through the ether. The waves radiating from a molecule will be shorter in the direction in which the molecule is moving than in the opposite direction. A multitude of molecules carrying similar sources of harmonic disturbance, but moving in different directions, will thus produce a great number of waves of lengths which vary slightly from that which would result if the molecules were stationary. The radiations emitted by a heated gas or vapour will not, therefore, be confined strictly to isolated wave-lengths, and the spectrum will not consist of mere lines, but of bands bright at the centre, and shading off at the edges. A rise of temperature will increase the breadths of the spectral lines.

This result can easily be observed, by the aid of a spectrometer which will separate the D lines. A Bunsen flame into which a little common salt has been introduced emits radiations which are approximately homogeneous, so that the spectral lines are narrow. If, however, metallic sodium is introduced into an electric arc, each spectral line is much broadened and blurred at its edges, and its centre is marked by a black line. The sodium molecules in the cool outer layer of vapour absorb the wave-lengths corresponding to their natural periods, and thus reverse the centre of each line; but the longer and shorter waves emitted by the quickly moving molecules in the intensely heated arc are allowed to pass, and produce the blurred edges of the lines observed.

[1] See *Heat for Advanced Students*, by the Author, p. 296.

## Questions on Chapter XIV

1. Give a general explanation of Fraunhofer's lines in the solar spectrum, and describe an experiment to verify the explanation.

2. What information as to the constitution of the heavenly bodies can be obtained from an examination of their spectra?

3. Sketch the plan of a spectroscope, explaining the use of the collimator; and describe how to make a map of the spectrum of a given substance.

4. The spectra of many gases consist of large numbers of very fine lines. What relations have been discovered between the frequencies of vibration which correspond to different lines in such spectra?

5. An iron ball is made white hot, and its spectrum examined by a spectroscope. What will be the nature of the spectrum seen? Again, a piece of iron is used as one of the poles in the electric arc, and the spectrum examined. In what respects do the two spectra differ from one another? To what molecular conditions do you suppose the difference to be due?

6. Give some account of the instrumental methods used by Langley in his investigation of the infra-red radiations of the sun.

7. Describe the bolometer, and explain the method of using it to investigate the infra-red portion of the spectrum.

8. How would you prove that the thermal, chemical, and luminous effects of the same part of the visible spectrum are not due to three different causes, such, for instance, as three different kinds of coincident rays?

9. Describe a method of investigating the infra-red part of the spectrum, and give the principal results arrived at.

10. If the earth were moving very rapidly through space, what would be the general effect on the spectra of stars which it was (1) approaching, (2) receding from? Give full reasons for your answer.

## Practical

1. Draw, for the given spectroscope, a curve showing the relation between the wave-length and the readings of the scale in the spectroscope.

2. Construct a map of the absorption spectrum of the given liquid, using light of known wave-lengths to calibrate the spectrometer.

3. You are supplied with specimens of salts, and with a mixture of several of them. Determine, by means of the spectroscope, which of the salts are present in the mixture.

4. Map the spectra of the given metallic salts.

5. Set the prism on the spectroscope to minimum deviation for soda light, and determine, for different thicknesses of the liquid supplied to you, the difference of the deviation of D and of the red end of the absorption band produced by the liquid. Exhibit your results in a curve, the thickness of the absorbing layer being abscissæ.

# CHAPTER XV

## RADIATION, ABSORPTION, AND DISPERSION

**Mechanical Pressure of Light.**—On the corpuscular theory, light should exert a mechanical pressure on a body on which it is incident. Each light corpuscle must possess energy, and, on striking a body, its velocity must be annulled (if the light is absorbed) or reversed (if the light is totally reflected). The pressure would be greater in the case of reflection than in that of absorption, from considerations similar to those used in explaining the pressure of a gas on the kinetic theory.

There appears to be no obvious reason why transverse waves in an elastic solid should produce a mechanical pressure; but Maxwell proved that, on his Electro-Magnetic Theory, sunlight should exert a pressure amounting to about 0·4 milligrams per square metre of a black surface, or about 0·8 milligrams per square metre of a perfectly reflecting surface, the light in both instances being incident normally. More generally, Maxwell's law states that the mechanical pressure per unit surface, due to a parallel pencil of light incident normally, is equal to the energy per unit volume of the ether near the surface. If there is a reflected ray, its energy must be added to that of the incident ray.

It is difficult to measure such a small pressure, and complications arise from the circumstance that, to avoid the effect of air currents, the experiment must be performed *in vacuo*. If the vacuum is not perfect, the residual gas will exert reactions on the surface on which the light is incident. This is due to the circumstance that absorption of the light raises the temperature of the surface, and the gas molecules rebound

from the latter with a greater velocity than that with which they strike it. This is the explanation of the action of Sir William Crookes's **radiometer** (Fig. 194). In this instrument four platinum vanes are mounted on a light framework which is pivoted on a fine needle-point, the whole being enclosed in a glass vessel which is highly exhausted. Each vane is blackened on one side, and polished on the other. When the instrument is exposed to light, rays are absorbed by the black, and reflected from the polished, surfaces. Each blackened surface experiences a rise of temperature which does not penetrate to the opposite polished face of the vane. The residual gas is, in its turn, heated; in the terms of the kinetic theory, the gas molecules, striking a hot, blackened surface, rebound with an augmented velocity; and, since action and reaction are equal and opposite, a pressure is exerted on the black surface. Since the opposite polished surface of the vane is at a lower temperature, the molecules striking th ence a smaller increas and the resulting pres be less. Thus, the vanes will revolve as if the blackened repelled by light to a greater extent than the polished surf

FIG. 194.—Sir William Crookes's Radiometer.

According to Maxwell's theory, the true pressur twice as great on a polished as on a blackened s same result would follow from the corpuscular either of these theories, the vanes of a Crookes's would revolve in the opposite direction to that g served, if the vacuum were made perfect.

It is only quite recently, by the labours of Lebedew, that Maxwell's views have been confirmed. In a glass globe 20 cms. in diameter a thin glass rod, H (Fig. 195), was suspended by a very fine glass filament. This rod carried two sets of vanes, $P_1$ and $P_2$, each set consisting of two discs of platinum 5 mms. in diameter, at a distance of about 2 cms. apart. Of each

pair, one disc was polished on both sides, while the other was blackened on both sides. The discs of one pair were 0·1 mm. in thickness, while those of the other pair were 0·02 mm. in thickness. The glass rod, H, carried a small mirror, M, by means of which any rotation of the vanes could be detected and measured.

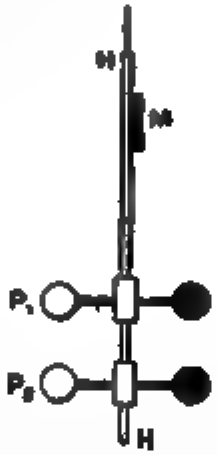

Fig. 195.—Suspended System in Lebedew's Experiment.

The arrangement of the apparatus is represented in Fig. 196. Light from an arc lamp, S, was rendered parallel by a lens, and then, by means of a set of mirrors and a second lens, was focussed on one of the discs, A. By moving the double mirror M from one side to the other, the light could be focussed first on one side, and immediately afterwards on the other side, of a particular disc. The mean of the two deflections thus produced will be independent of currents in the residual gas. The radiometer action, which depends on the *difference in temperature* of the opposite faces of a disc, would naturally be greater for a thick than for a thin disc. To render the radiometer action as small as possible, the exhaustion of the bulb was carried on until a greater deflection was produced when the light was incident on a polished vane than when it was incident on a blackened vane. After reducing his observations, Lebedew found that *light exerts a true pressure on a surface on which it is incident, this pressure being twice as great for a reflecting as for an absorbent surface. The absolute magnitude of the pressure was found to be equal to that predicted by Maxwell.*

Fig. 196.—Lebedew's Apparatus for measuring the Mechanical Pressure of Light.

**Tails of Comets.**—The above result is of interest, as it explains why a comet develops a tail as it approaches the sun. The tail is always directed away from the sun, no matter what may be the direction of the comet's motion (Fig. 197). This indicates a repulsion by the sun, and it follows from the law of the pressure of light, that such a repulsion would occur. The gravitational attraction of the sun on a body at a given distance from it, is proportional to the mass of the body, and this in its turn is proportional to the cube of the linear dimensions of the body. The repulsion, due to the pressure of light, will be proportional to the surface of the body, *i.e.* to the square of its linear dimensions. Hence, the ratio of the repulsion to the attraction will be inversely proportional to the linear dimensions of the body, so that in the case of a very small body the repulsion may preponderate. Thus, the tail of a comet probably consists of small stones, from a centimetre in diameter downwards, while the head of the comet consists of an aggregation of large stones. This explanation seems to have been first proposed by the late Prof. Fitzgerald.

Fig. 197.—Forms assumed by a Comet in its passage round the Sun.

**Fluorescence.**—As a general rule, when light is absorbed by a body, its energy becomes transformed into that of molecular motion, or heat. In certain cases the absorbed light gives rise to new light-waves, generally of a particular wave-length, or at least confined between comparatively narrow limits in the spectrum. This phenomenon is termed **fluorescence.** As a general rule, the violet and ultra-violet parts of the spectrum are most active in producing fluorescence. Sir George Stokes investigated this point as follows. He formed a pure spectrum on a screen, using sunlight or the electric arc as an illuminant, the prism and lenses employed being of quartz. A strip of white card, which had been painted thickly with a paste made from

sulphate of quinine moistened with a little dilute sulphuric acid, was placed beyond the violet end of the spectrum ; it was found to fluoresce brightly, emitting a blue light. A number of dark bands, similar to the Fraunhofer lines in the visible spectrum, were observed. The visible parts of the spectrum were practically inactive in producing fluorescence in the sulphate of quinine.

There is a great number of substances which exhibit fluorescence. Ordinary paraffin oil when exposed to sunlight exhibits a bluish fluorescence in the layer on which the light is incident. The reason why the fluorescence is confined to the layer on which the light is incident is, that this layer absorbs the waves which are capable of producing fluorescence, the transmitted light thereby being rendered inactive. One of the most brilliantly fluorescing substances is the aniline derivative termed *fluorescene*. This emits a brilliant yellowish-green light when exposed to daylight or the light from an electric arc lamp. If a piece of paper which has been moistened with a solution of fluorescene is placed on the surface of water in a large beaker, a beautiful tree-like growth, fluorescing brilliantly, will be observed to spread downwards from it ; each "twig" ends in a small vortex ring. A piece of crushed horse-chestnut bark, which contains a fluorescent substance termed *æsculin*, can be substituted for the paper moistened with fluorescene ; in this case the fluorescence is blue.

From his experiments, Sir George Stokes was led to frame the following law : **When the refrangibility of light is changed by fluorescence, it is always lowered, and never raised.** In other words, the waves emitted during fluorescence are always longer than those which are absorbed, and thus give rise to the fluorescence. Certain exceptions to this rule occur ; these may be explained by assuming that, in cases where Stokes's law is not obeyed, some sort of chemical reaction occurs.

Sodium vapour fluoresces brilliantly when exposed to sunlight, emitting rays which correspond to bands in the green and red parts of the spectrum, together with a very bright yellow band in the mean position of the D lines.

**Phosphorescence.**—Fluorescence continues only so long as light is incident on the fluorescent substance. Certain substances, after being exposed to light of short wave-lengths, continue to emit light when placed in a dark room. This

phenomenon is termed **phosphorescence.** Prominent amongst phosphorescent substances are the sulphides of calcium, barium, and strontium. Balmain's luminous paint is composed of these sulphides. It will continue to phosphoresce for some hours in a dark room after exposure to sunlight.[1]

It is found that violet and ultra-violet light are most active in producing phosphorescence. If a card, coated with Balmain's paint and made slightly luminous by a short exposure to sunlight, is then exposed for some time to a continuous spectrum, it is found on removing the card to a dark room, that the parts illuminated by the less refrangible rays of the spectrum have ceased to phosphoresce. Thus, the incidence of long waves on a phosphorescing body tends to destroy the phosphorescence.

Becquerel found that many substances, which apparently are not phosphorescent, yet emit visible radiations for a short

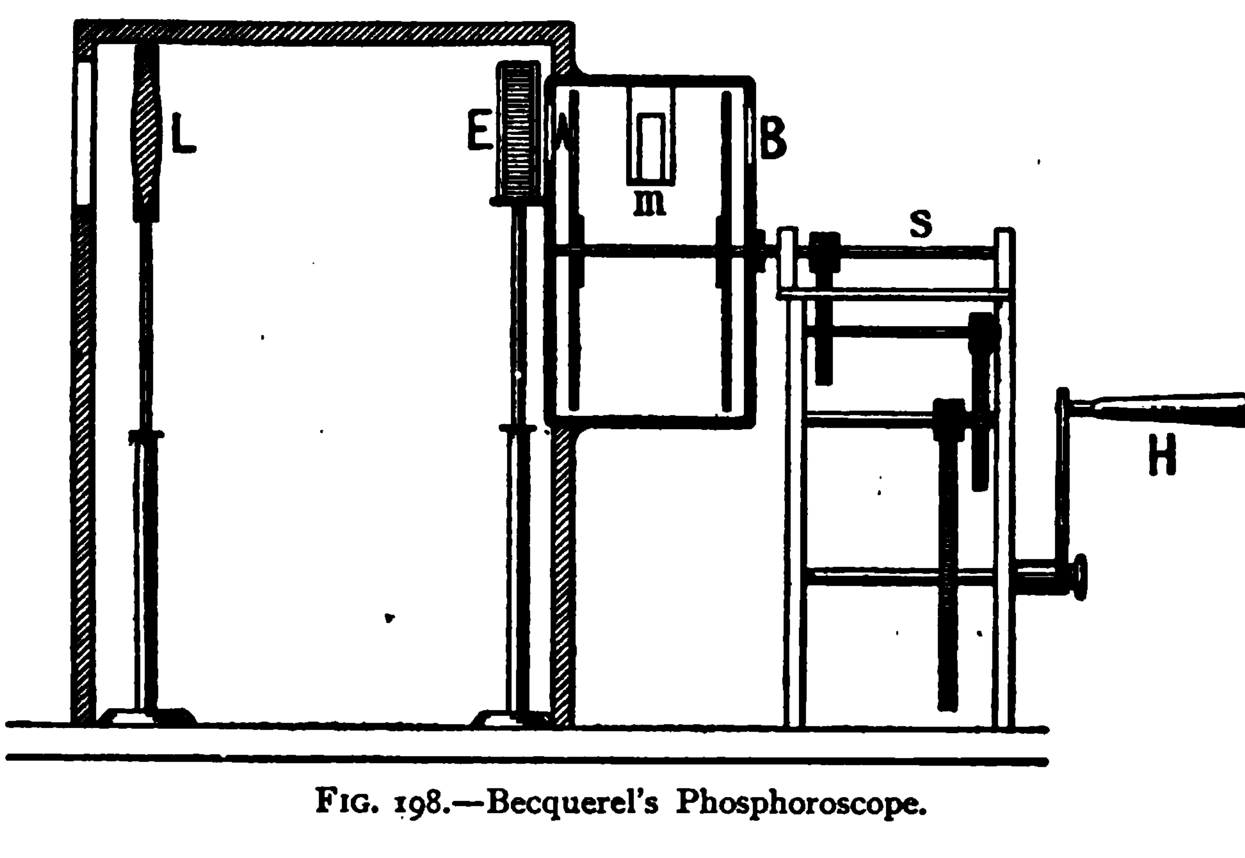


FIG. 198.—Becquerel's Phosphoroscope.

time after exposure to light. The apparatus he used, termed a **phosphoroscope,** is represented in Fig. 198. Sunlight, after traversing the lens L, is brought to a focus at *m*. The glass tank E, placed in the path of the light, can be filled with a solution which absorbs any particular wave-length which it may

[1] Exposure of a substance to sunlight in order to provoke phosphorescence, is often termed *insolation.*

be required to suppress. The substance to be tested for phosphorescence is supported by a stirrup at *m*, inside a cylindrical box, provided with two opposite windows, A and B, in its plane ends. The axis of the cylindrical box is traversed by a spindle, S, which carries two metal discs, the latter being pierced at intervals with circular apertures which pass across the windows A and B as the discs rotate. The discs are adjusted so that the apertures in one are midway between those in the other; thus, as the discs rotate, the windows A and B are alternately covered and uncovered, B always being covered when A is uncovered, and *vice versâ*. The discs and the interior of the box are coated with dull black paint. The spindle, S, is set in rapid rotation by means of a handle, H, actuating a train of toothed wheels. Arrangement is made for determining the speed of rotation.

It now becomes easy to understand the method of using the phosphoroscope. As the discs are caused to rotate, an aperture in the left-hand disc will come in front of the window A, and expose the substance at *m* to the sunlight which has traversed the lens L and the tank E. At this instant the window B is covered by the second disc; but after a short interval of time, dependent on the speed at which the discs are rotating, the window A will be covered and the incident light cut off, while the window B will be uncovered. An observer looking at the window B will thus see the substance at *m* only when it is not illuminated by sunlight; by varying the speed of rotation of the discs the interval between an illumination of *m* by sunlight, and its exposure to the view of the observer after the sunlight has been cut off, can be varied at will. If the experiment is performed in a dark room, the substance at *m* will not be seen unless it phosphoresces. If the substance is phosphorescent, the persistence of visual impressions will allow the observer to examine the nature of the radiations emitted; the emitted light may be allowed to fall on the slit of a spectrometer, so that the wave-length of the phosphorescent light can be determined.

The following are some of the conclusions reached by Becquerel:—

(1) After exposure to light, a body may phosphoresce for a period varying between 1/5000 second and several hours, according to the nature of the body. The strongest and most enduring phosphorescence

is exhibited by compounds of the alkalis and alkaline earths, together with a few metallic salts. Compounds of alumina are very active, while those of silica are quite inactive.

(2) The phosphorescence occurs throughout the volume of the body, if that is small; it depends only on the intensity and refrangibility of the incident light.

(3) With the same body the colour of the light emitted varies with the time which has elapsed after exposure to light. Thus, when examining the diamond, the phosphorescence is yellow or orange when the discs are rotating slowly; when the speed of the discs is considerably increased, the tint of the emitted light becomes blue.

(4) The spectrum of the emitted light consists in all cases of bands of greater or smaller width, and in all cases the wave-length of the emitted light is either greater than, or equal to, that of the incident light. The same body may emit radiations of different wave-lengths when excited by light of different wave-lengths.

Professor Dewar has found that many substances, such as feathers, egg-shells, etc., acquire the power of phosphorescing brilliantly, on being cooled to the temperature of liquid air.

**Calorescence.**—When light-waves are absorbed at a black surface, their energy is converted into heat, or energy of molecular motion. The molecules of the absorbing substance, being set in motion, become sources of disturbance in the ether, and generate waves in the latter. These waves, in general, are too long to be perceived by the eye; but under special conditions visible radiations may be emitted. Tyndall found that a solution of iodine in carbon-bisulphide is completely opaque to waves corresponding to the visible part of the spectrum, but is transparent to a great proportion of the long infra-red waves. Accordingly, he placed a thin spherical glass flask, containing the solution mentioned, in front of an arc lamp, when the transmitted infra-red rays were brought to a focus, the light-rays being absorbed. A piece of paper or a cigar placed at this focus immediately burst into flame; a piece of very thin blackened platinum foil was raised to a white heat. In the latter case the spectrum of the light emitted was continuous, possessing a point of maximum energy for a certain wave-length, just as in the case of light emitted by platinum heated by any other means. Of course, it is impossible, by any arrangement of lenses or mirrors, to raise the temperature of an absorbing body above

that of the source; unless, indeed, some sort of chemical change, in which energy is liberated, is induced. This result follows from the Second Law of Thermodynamics. Hence, luminosity cannot be produced by absorbing radiations from a source which itself is not hot enough to be luminous.

It has sometimes been stated that calorescence is the inverse of fluorescence. The two phenomena do not appear to be strictly comparable. Fluorescence is produced by the selective absorption of waves of particular lengths, while in calorescence the absorption is general. In fluorescence the radiations emitted are comprised in a limited number of spectral bands; in calorescence radiations corresponding to a continuous spectrum are emitted. Thus, the two phenomena appear to be due to entirely distinct causes. In fluorescence only the vibrating particles (or electrons) which possess certain free periods are set in motion, and this motion is of a regular or periodic nature. In calorescence the molecules themselves are set in motion and the violent and irregular motion of the electrons, produced by collisions or some similar cause, results in the emission of arbitrary disturbances, which, on being analyzed by a prism, are resolved into an infinite number of harmonic waves, lying between certain limits in the complete spectrum.

**X Rays.**—One of the most interesting discoveries of the last century was that of the radiations which their discoverer, Professor Röntgen, termed "X rays," but which are now frequently termed "Röntgen rays." A general description of the manner in which these radiations are produced, together with their properties, will now be given; it will then be possible to form a judgment as to their probable relation to light.

For the production of X rays, a vacuum tube, of a shape similar to that represented in Fig. 199, is used. The electrode K, which consists of a small concave spherical mirror made from thin sheet aluminium, is connected, by means of a platinum wire sealed into the walls of the tube, to the negative terminal of a powerful induction coil. The other electrode, A, which consists of a small sheet of platinum foil inclined at an angle of 45° to the axis of symmetry of the tube, is connected in a similar manner to the positive terminal of the induction coil. The intermittent current thus enters the tube at A, and leaves it at K. A is termed the *Anode*, and K the *Kathode*, of the tube.

As the tube is exhausted, some very remarkable changes occur.

When the enclosed gas is at atmospheric pressure, the discharge takes the form of a spark, narrow and tortuous in its course. As exhaustion proceeds, the spark spreads out laterally into a luminous brush, which nearly fills the tube. By degrees the luminous brush becomes stratified, and a dark space appears in the immediate neighbourhood of the kathode. As the exhaustion proceeds, this dark space increases in magnitude; when it has enlarged so far as to reach to the glass walls of the tube, these latter become phosphorescent. Within the dark space can be seen faint blue streamers; these leave the

FIG. 199.—"X Ray" Tube.

kathode, K, normally, and converge toward its centre of curvature, at which point A is situated. These blue streamers are termed **Kathode Rays.** They apparently consist of streams of negatively charged particles of very small mass, but moving with considerable velocity. Wiechert has directly determined the velocity of the particles producing the kathode rays, and finds this to be about $5 \times 10^9$ cms. per second. Prof. J. J. Thomson estimates that the mass of each particle lies between 1/500 and 1/1000 of the mass of a hydrogen atom. Consequently, it appears that atoms can be split up into simpler constituents.

The properties of the kathode rays are as follow:—

1. They usually travel in straight lines, but if a magnet is placed in their neighbourhood, they are deflected, and assume a spiral path round the lines of magnetic force. If flexible conductors carrying electric currents occupied the same positions as the kathode rays, the conductors would curl round the lines of force in a similar manner. It has been established, theoretically, that the path of a charged particle

is modified by a magnetic field in the same manner as if it were a flexible conductor carrying an electric current. Thus, the magnetic deflection of the kathode rays is taken as proof that these consist of streams of charged particles. Experiment proves that their charges are negative.

2. When incident on a movable object, kathode rays may set the latter in motion. The molecules of a body are also disturbed, since a piece of platinum foil may be raised to a white heat, or even melted, under the action of kathode rays.

3. Glass is opaque to the kathode rays, but is caused to fluoresce brightly under their impact. Soda-glass emits a yellow-green light, which shows a band coinciding with the D lines when spectroscopically examined; lead-glass emits a blue light; while diamonds, precious stones, and the rare earths emit light-radiations of characteristic colours.

4. A thin sheet of aluminium foil is practically transparent to kathode rays. By making a small aluminium window in the side of a vacuum tube, Lenard succeeded in leading the kathode rays into the open atmosphere. At atmospheric pressure, they can penetrate a layer of air 4 or 5 cms. thick. They affect photographic plates, and produce phosphorescence in many substances, notably in barium platino-cyanide. They can still be deflected by a magnet.

Soon after the exhaustion of the vacuum tube has been carried to the point at which the dark space surrounding the kathode, K (Fig. 199), extends to the anode, A, a remarkable change in the appearance of the tube ensues. If we imagine a plane to be drawn through the anode A, then the whole of the walls of the tube on the kathode side of this plane become brightly fluorescent. The appearance of the tube at this stage is represented in Fig. 199. It follows that the radiations causing this fluorescence are emitted by the platinum anode A; they are not given off solely in a normal direction, as the kathode rays are, but are emitted in all directions like the light from a piece of white-hot platinum foil. If the platinum anode A is shaded from the kathode rays, the radiations we are discussing are no longer given off from it. Any body interposed in the path of the kathode rays gives off similar radiations, whether it forms the anode or not.

The radiations, emitted by a body placed in the path of the kathode rays, can escape through the walls of the vacuum tube (if these are of soda-glass), and can penetrate a layer of air several feet thick. They then constitute the X rays discovered by

Professor Röntgen. They travel in straight lines, and produce bright fluorescence in many substances, such as barium platino-cyanide. They also blacken silver salts, and affect a photographic plate very much as violet or ultra-violet light does. Their most remarkable property, however, is that they are freely transmitted through many bodies which are entirely opaque to light. It has been proved that the opacity of a substance to X rays is simply proportional to its density. Thus wood, soda-glass, and aluminium, which are substances possessing very small densities, are practically transparent to X rays. Lead and platinum, and the denser metals are, comparatively speaking, opaque to X rays. If a body opaque to X rays is placed between the vacuum tube (Fig. 199) and a photographic plate, a shadow radiograph of the body is obtained. Since wood is transparent to X rays, it is unnecessary to open the dark slide in which the photographic plate is contained. Flesh is comparatively speaking transparent to X rays, while bone, being much denser, is more opaque. Consequently, if the hand is placed between an X ray tube and a photographic plate contained in a dark slide, a shadow radiograph of the bones of the hand can be obtained. A needle or other metallic body embedded in the flesh or bone casts a distinct shadow. In the accompanying radiograph (Fig. 200) of a human hand, the dark band around the little finger represents a ring. The break in the ring is due to a stone through which the rays were able to pass. Photographs of the ribs and backbone of living persons have also been obtained.

If a cardboard screen, coated on one side with a fluorescent substance, such as barium platino-cyanide, is placed in front of an X ray tube, so that the fluorescent side of the screen faces the observer, the light emitted by the tube is cut off, while the fluorescent substance is energetically acted upon, since the cardboard is transparent to the X rays. A hand held between the tube and the screen casts a perfectly definite shadow; the flesh casts a faint shadow, while the shadows of the bones are darker.

X rays cannot penetrate a very great thickness of air. They appear to become diffused, as light does when transmitted through a layer of milk and water.

**Properties of X Rays.**—1. **X rays are not refracted** by material media; a glass prism interposed in their path does not produce any deviation.

2. **X rays are diffusively reflected** from a polished surface, much as light is reflected from ground glass. There is no definite reflected ray corresponding to a particular incident ray.

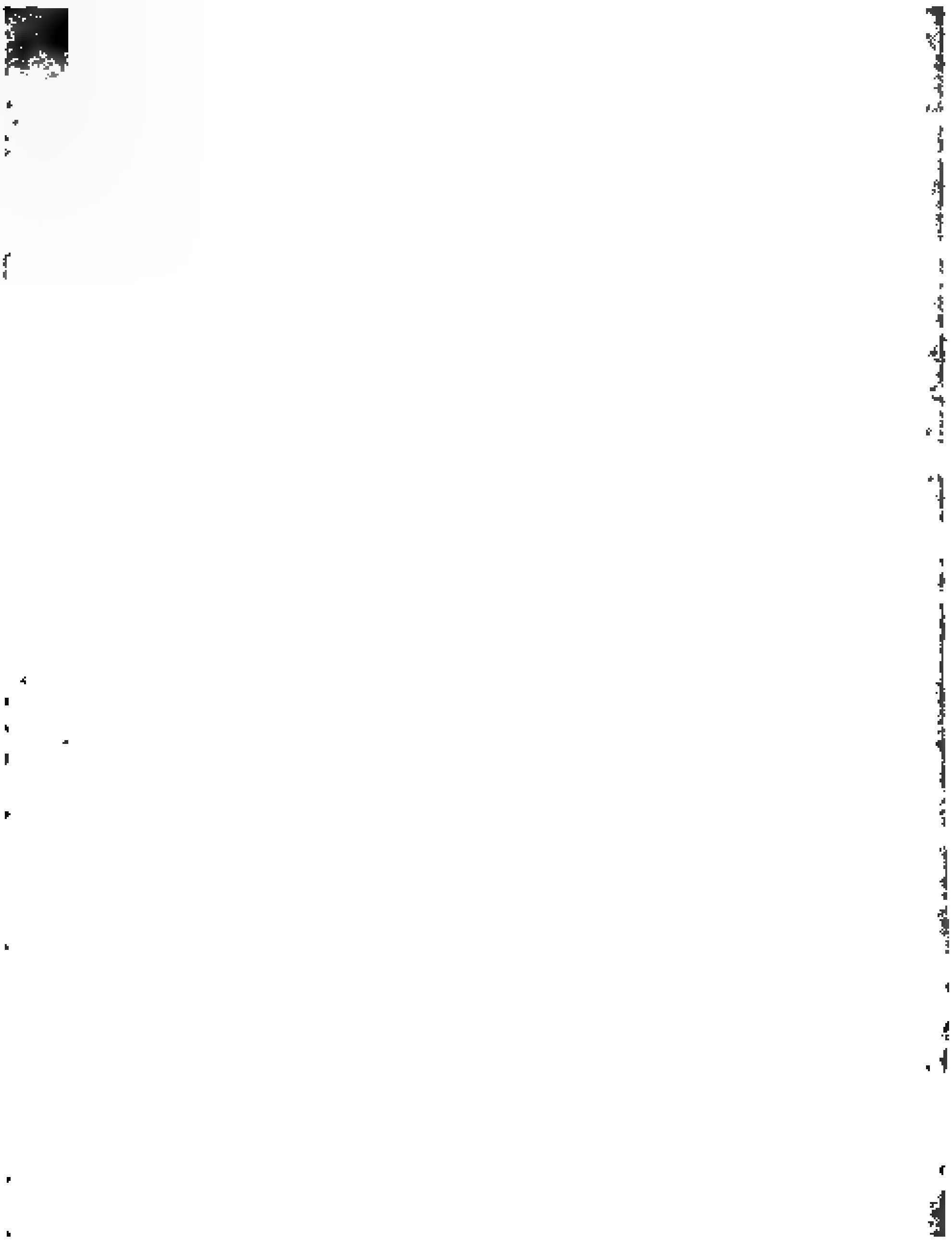

FIG. 200.—Radiograph of Human Hand.

3. **X rays are not polarised** by material media. No extinction is produced by crossed tourmalines (compare p. 324).

4. **The velocity of transmission of X rays** has not been definitely determined, but it has been proved to be very great, comparable with that of light.

5. No indisputable evidence of **interference** has been obtained with respect to X rays.

6. **X rays are not deflected by a magnet,** as kathode rays are. Thus X rays do not consist of streams of charged particles.

7. **X rays, on passing through air, cause the latter to conduct electricity.** Ultra-violet light has a similar action. It appears that in both cases negatively charged particles, each possessing a mass comparable with 1/1000 of the mass of a hydrogen atom, are shaken off from the air molecules.

**Theory of X Rays.**—It appears probable that X rays consist of ultra-violet light of extremely short wave-length. This would account for the difficulty of obtaining interference, since the bands may be too narrow to be visible. It would also explain the diffusive reflection of X rays. The smoothness of a surface necessary to obtain regular reflection depends on the wave-length of light employed; ground glass, which reflects the long infra-red rays regularly, reflects ordinary light diffusively. We shall see presently that Sellmeier has proposed a theory of refraction, according to which all material media should possess a refractive index equal to unity for radiations of very short wave-length. This will account for the absence of refraction and polarisation.

**Becquerel Rays.**—Röntgen discovered the X rays in 1895. In 1896 Becquerel discovered that the salts of uranium emit invisible radiations which affect photographic plates and cause atmospheric air to become a conductor of electricity. This property is not limited to the fluorescent uranic salts, but is shared by the non-fluorescent uranous salts. The emission of these radiations has been observed to continue, without any apparent diminution, during three years, in which time the active substance was preserved in a leaden box with double walls.

A Polish lady, Madame Curie, has found that pitchblende, the mineral from which salts of uranium are extracted, contains a substance apparently allied to bismuth, which emits Becquerel rays with 4000 times the activity of uranium. This substance, which has not been completely isolated, has been termed *polonium*, in honour of the native land of the discoverer. A second similar substance, allied to barium, has been found in pitchblende; this has been termed *radium*. A third substance, termed *actinium*, has also been found; this latter

substance appears to be allied to thorium. All three substances continuously emit radiations which affect photographic plates.

Becquerel rays can be neither reflected, refracted, nor polarised. They have been proved to comprise three distinct kinds of radiation. The $\alpha$ rays consist of streams of positively charged particles, of atomic dimensions, travelling with a velocity of $2 \times 10^9$ cms. per second. The $\beta$ rays consist of streams of negatively charged particles, each much smaller than an atom, and travelling with a velocity of $1 \times 10^{10}$ cms. per second. The $\gamma$ rays are apparently similar to X rays. It has been estimated by Rutherford that a given quantity of radium loses about 1/30,000 of its mass per annum in the form of $\alpha$ rays, which appear to consist of helium atoms.

**Sellmeier's Theory of Dispersion.**—We are now in a position to appreciate a beautiful theory, proposed by Sellmeier in 1871, to account for the refraction and dispersion of light by material media. We have already seen that it must be supposed that material substances are interpenetrated by the ether; each material molecule is surrounded by the ether, much as a leaf of a tree is by the air. In order to account for refraction and dispersion, it must be assumed that the properties of the ether are modified by the presence of the material molecules, in such a manner that the velocity of wave transmission through it is diminished. Some investigators have assumed that the material molecules condense the ether in their neighbourhood. This assumption presents two serious difficulties. In the first place, to account for the absence of longitudinal waves, it is generally assumed that the ether is incompressible. In the second place, a true condensation of the ether would entail an equal velocity for all wave-lengths of light, which is contrary to our experimental knowledge.

Sellmeier proceeded on other lines. He assumed that the material particles which, by their vibrations, produce light-waves, possess definite positions of equilibrium with regard to the ether; thus, any displacement of the ether in the neighbourhood of a molecule displaces the position of equilibrium of the vibrating particle. On the other hand, a particle when displaced with respect to the ether, oscillates to and fro about its position of equilibrium in a definite period. The number of vibrating particles is supposed to be so great that a considerable number lie along a length equal to that of a light-wave; but the

particles are themselves so small that no appreciable amount of the ether is displaced to make room for them, and the rigidity of the ether is unmodified by their presence. In this case, light-waves travelling through a material substance will set the vibrating particles in motion, and the reactions of these will modify the velocity of wave transmission.

We have already (p. 280) obtained a formula for the velocity of wave transmission through an elastic solid in which heavy particles, capable of free vibrations, are embedded. In a medium of which unit volume encloses $n_1$ particles with a free vibration period equal to $T_1$, together with $n_2$ particles with a free vibration period equal to $T_2$, waves of period T will be transmitted with a velocity V, given by the equation—

$$\frac{V_0^2}{V^2} - 1 = n_1 K_1 \frac{T^2}{T^2 - T_1^2} + n_2 K_2 \frac{T_2}{T^2 - T_2^2}, \quad . \quad . \quad (1)$$

where $V_0$ is the velocity of wave transmission in the medium when unhampered by vibrating particles, and $K_1$ and $K_2$ are constants depending on the dynamical properties of the particles (p. 281).

We can now simplify this formula. In the first place, the ratio $V_0/V$ will be equal to the refractive index, $\mu$, of the medium. Further, we may multiply the numerators and denominators of the fractions to the right of (1) by $V_0^2$. Then $V_0T$ will be equal to $\lambda$, the length, in the free ether, of the waves of period T. We may substitute $\lambda_1$ for $V_0T_1$; this is the length, in the free ether, of waves with a period $T_1$, equal to that of one set of vibrating particles; in other words, $\lambda_1$ is the wave-length of the radiations which the particles will emit when set in vibration. Similarly, we may write $V_0T_2 = \lambda_2$. Then—

$$\mu^2 = 1 + n_1 K_1 \frac{\lambda^2}{\lambda^2 - \lambda_1^2} + n_2 K_2 \frac{\lambda^2}{\lambda^2 - \lambda_2^2}, \quad . \quad . \quad (2)$$

which determines the refractive index of the medium in terms of the wave-length.

Equation (2) was originally obtained by Sellmeier. It is often, unjustly, termed the Ketteler-Helmholtz dispersion formula. The formulæ obtained by Ketteler and Helmholtz differ from (2), and the work of these investigators was subsequent to that of Sellmeier.

**Interpretation of Sellmeier's Dispersion Formula.**—We must now examine the manner in which the refractive index of

a substance will vary with the wave-length of the incident light.

1. INFINITELY SHORT WAVES.—If $\lambda$ is excessively small in comparison with $\lambda_1$ and $\lambda_2$, the two fractions to the right of (2) will be very small in value, and we may equate them to zero. In that case—

$$\mu^2 = 1.$$

Thus, **infinitely short waves will be transmitted with a velocity equal to that in the free ether.** If X rays are ether waves, similar to those of ordinary light, but of very short length, the failure of material media to refract them is accounted for.

After Professor Röntgen's discovery, it was frequently pointed out that, according to the dispersion formula obtained by Helmholtz, very short waves should not be refracted by material media. It seems to have escaped general notice that the same result was predicted by Sellmeier.

2. INFINITELY LONG WAVES.—We may re-write (2) as

$$\mu_2 = 1 + n_1K_1\frac{\lambda^2 - \lambda_1^2 + \lambda_1^2}{\lambda^2 - \lambda_1^2} + n_2K_2\frac{\lambda^2 - \lambda_2^2 + \lambda_2^2}{\lambda^2 - \lambda_2^2}.$$

$$\therefore\ \mu^2 = 1 + n_1K_1 + n_2K_2 + n_1K_1\frac{\lambda_1^2}{\lambda^2 - \lambda_1^2} + n_2K_2\frac{\lambda_2^2}{\lambda^2 - \lambda_2^2} \quad (3)$$

When $\lambda$ is very large in comparison with $\lambda_1$ and $\lambda_2$, the two fractions to the right of (3) will be negligibly small. Thus, if $\mu_\infty$ is written for the refractive index of a substance for waves of infinite length, we have—

$$\mu_\infty^2 = 1 + n_1K_1 + n_2K_2.$$

From the value of K given on p. 280, it will be seen that $n_1K_1$ and $n_2K_2$ are essentially positive quantities; hence **the refractive index of a material substance for infinitely long waves will always be greater than unity.**

3. DISPERSION.—Equation (3) may be re-written—

$$\mu^2 = \mu_\infty^2 + \frac{C_1}{\lambda^2 - \lambda_1^2} + \frac{C_2}{\lambda^2 - \lambda_2^2},$$

where $C_1 = n_1K_1\lambda_1^2$, and $C_2 = n_2K_2\lambda_2^2$. For a given medium in a given physical state, $C_1$ and $C_2$ will be constants.

The manner in which $\mu$ varies with $\lambda$ is shown in Fig. 201.

To obtain the curves in Fig. 201 it was assumed that $\mu_x{}^2 = 1{\cdot}7$, while $\lambda_1 = 12$, and $\lambda_2 = 6$. The value assumed for $C_1$ was equal to $43{\cdot}2$, while $C_2 = 14{\cdot}4$. In this case—

$$\mu^2 = 1{\cdot}7 + \frac{43{\cdot}2}{\lambda^2 - 144} + \frac{14{\cdot}4}{\lambda^2 - 36} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Substituting various values, between 12 and 25, for $\lambda$ in (4), it is seen that as $\lambda$ diminishes, the denominators of the fractions to the right

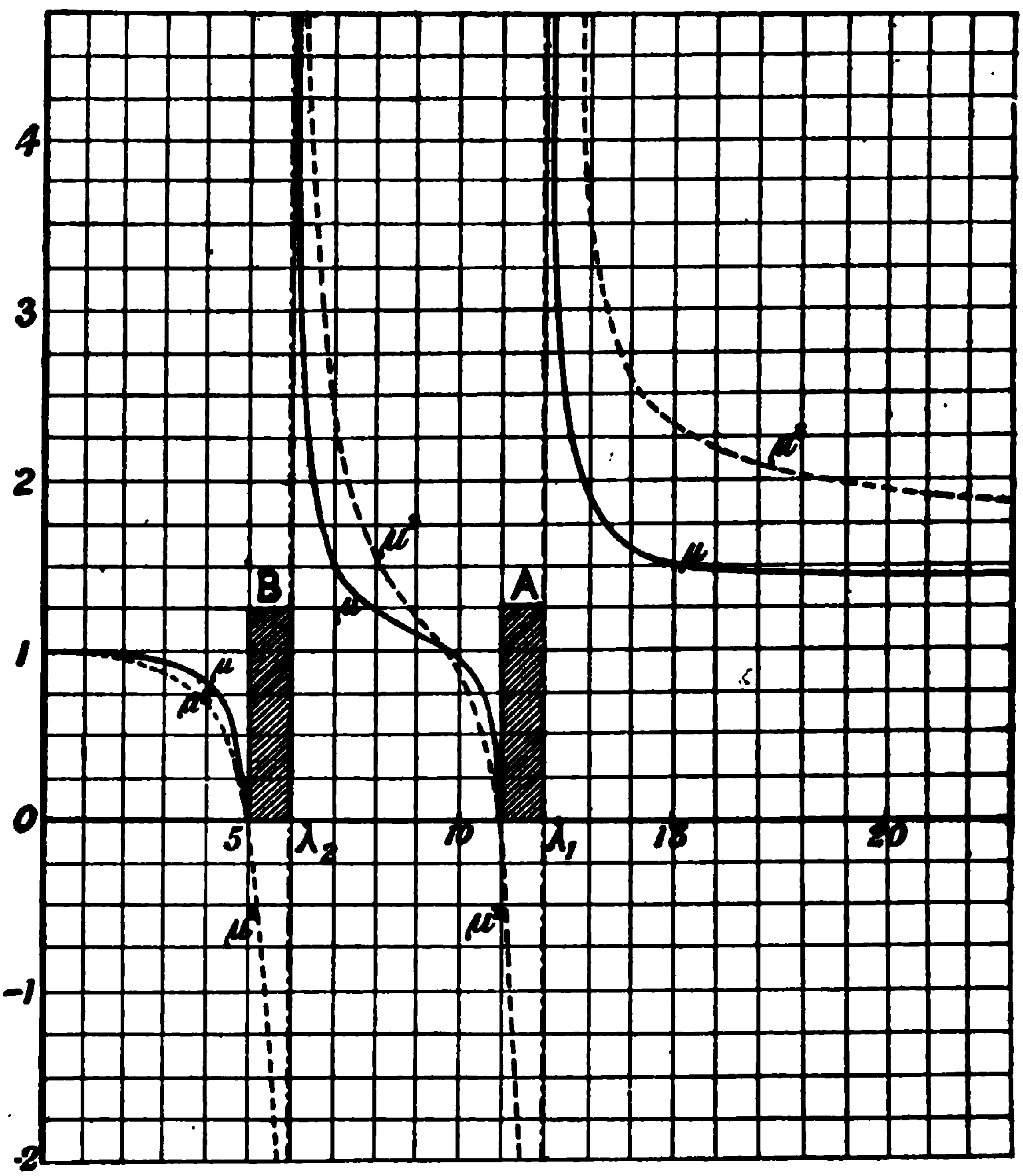


FIG. 201.—Graphical Interpretation of Sellmeier's Dispersion Formula.

of (4) decrease, so that the value of $\mu^2$ increases. As the value of $\lambda$ approaches 12, the denominator of the fraction

$$\frac{43{\cdot}2}{\lambda^2 - 144} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

approaches zero, and thus the value of the fraction itself increases

indefinitely. Thus, the broken line curve for $\mu^2$ (Fig. 201) bends sharply upwards for values of $\lambda$ a little greater than $\lambda = 12$, and would touch the straight line $\lambda = 12$ at infinity. The full-line curve for $\mu$ was found by extracting the square roots of various ordinates to the curve for $\mu^2$, plotting the results above the corresponding values of $\lambda$, and joining the points so found.

Now, $\lambda_1 = 12$ is the wave-length of the radiations which the medium would emit if the particles, of which the free period is $T_1$, were set in vibration. Hence, by Kirchhoft's law (p. 339), the medium will possess an absorption band for $\lambda = 12$. We thus see that **the medium will possess an abnormally high refractive index for waves slightly longer than those which it absorbs.**

As the value of $\lambda$ is changed, from one slightly greater than 12 to one slightly less than 12, the denominator of the fraction (5) changes from a very small positive to a very small negative quantity. The fraction (5) itself changes from an indefinitely great positive, to an indefinitely great negative, quantity. Thus, it is seen that for values of $\lambda$ slightly less than 12, the value of $\mu^2$ is negative. As $\lambda$ decreases, the fraction (5) remains negative, but its numerical value decreases. On the other hand, the fraction

$$\frac{14{\cdot}4}{\lambda^2 - 36} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

possesses a positive value which increases as $\lambda$ approaches the value 6. Consequently, as $\lambda$ is diminished from 12 to 6, the value of $\mu^2$ changes from an infinitely great negative, to an infinitely great positive, value. Before the curve for $\mu^2$ crosses the axis (*i.e.* when $\mu^2$ is negative) we can find no real value of $\mu$, and there can therefore be no transmission of light. The shaded rectangle A (Fig. 201) thus represents the position of an absorption band which is not confined to the wave-length corresponding to that of the free period of the vibrating particles, but extends some distance toward the violet end of the spectrum. The breadth of this band will, however, be small, unless the number of vibrating particles per unit volume is very great. Becquerel has observed a broadening of the absorption bands due to sodium vapour, when the density of the latter is increased.

After the curve for $\mu^2$ crosses the axis, $\mu^2$ acquires a positive value, which is at first less than unity. The corresponding wave-lengths of light are transmitted through the medium with a velocity greater than that in the free ether. Thus, **the medium will possess an abnormally low refractive index for waves slightly shorter than those which it absorbs.**

It will readily be seen that as $\lambda$ passes through the value 6, the value

of $\mu^2$ once more changes from an infinite positive to an infinite negative quantity. Finally, as $\lambda$ decreases from 6 to 0, the value of $\mu^2$ changes from a negative to a positive value; this positive value approaches unity as $\lambda$ is diminished. The shaded rectangle B represents the position of an absorption band, corresponding to values of $\lambda$ for which $\mu^2$ is negative.

The curves (Fig. 201) indicate that in general the refractive index of a substance increases as the wave-length of the incident light diminishes. Just below an absorption band (*i.e.* on the side of it toward the red) the refractive index will possess an abnormally high value. Just above an absorption band (*i.e.* on the side toward the violet) the refractive index will be abnormally small, even less than unity.

**Verification of Sellmeier's Theory.**—Le Roux found that the vapour of iodine refracts red more strongly than violet light, and in 1870 Christiansen announced the result that for an alcoholic solution of the aniline dye fuchsine (often termed magenta), the refractive index increases from the Fraunhofer line B to D, then sinks rapidly as far as G, and increases again beyond. The experimental investigation of the subject was continued by Kundt, who proved that this **anomalous dispersion** (as it is termed) is marked in all substances possessing strong surface-colours. As the result of his experiments, Kundt was able to lay down the rule that **in going up the spectrum, from red to violet, the deviation is abnormally increased below an absorption band, while above the band the deviation is abnormally diminished.**

Pfluger has recently succeeded in constructing small acute-angled prisms of solid fuchsine. On examining the transmitted light, he confirmed Kundt's results, and in addition proved that for $\lambda = 4500$ tenth-metres, the value of $\mu$ is less than unity; while for $\lambda = 6000$ tenth-metres (in the neighbourhood of the D lines), $\mu = 2{\cdot}6$. Fuchsine strongly absorbs green light, so that $\lambda = 6000$ will be a wave-length slightly longer than those absorbed, while $\lambda = 4500$ will be a wave-length slightly smaller than those absorbed.

Fig. 202 is reproduced from a spectrum photograph obtained by Prof. R. W. Wood. The horizontal band above VB . . . R represents an ordinary continuous spectrum produced by using a glass prism with refracting edge vertical. A small solid prism

of cyanine (an aniline dye possessing a strong absorption band in the greenish-yellow, yellow and orange) was placed, with its refracting edge horizontal and downwards, in the path of the spectral rays. The resulting spectrum is seen above. The red light is strongly refracted upwards (toward the base of the cyanine prism). The orange, yellow, and greenish-yellow rays have been absorbed, the bluish-green light is slightly refracted, and the refraction increases from the blue to the violet. (Compare p. 84.)

FIG. 202.—Illustrates the Anomalous Dispersion of Cyanine.

One of the most remarkable verifications of Sellmeier's theory is afforded by Fig. 203, which is reproduced from a spectrum photograph obtained by Becquerel. In the path of rays forming a horizontal continuous spectrum, a wedge-shaped flame, strongly coloured with sodium, was placed. This flame acted as a prism of sodium vapour with refracting edge horizontal and upwards. In its general course the spectrum is slightly displaced upwards, due to the small density of the gases in the flame; this displacement is seen from the position of the horizontal black line, which would bisect the continuous spectrum longitudinally if the sodium flame were absent. The red end of the spectrum is to the left. Immediately to the left of the position occupied by the line $D_1$, the spectrum curves sharply downwards (*i.e.* toward the base of the wedge-shaped flame), indicating an abnormal increase in the refractive index of sodium vapour for wave-lengths slightly greater than that of $D_1$. Immediately

FIG. 203.—Anomalous Dispersion of Sodium Vapour.

to the right of $D_1$ the spectrum is deviated upwards (toward the refracting edge of the wedge-shaped flame), indicating an abnormally small refractive index for wave-lengths slightly less than that of $D_1$; the photograph indicates that for these waves the refractive index of the sodium vapour must have been considerably less than unity. As the wave-length decreases, the upward deviation diminishes, and finally gives place to a strong downward deviation in the neighbourhood of $D_2$. Above $D_2$ we once more have an upward deviation, which rapidly diminishes as we proceed along the spectrum. Thus, Kundt's law, which follows as a matter of course from Sellmeier's dispersion theory, is exemplified in the refraction of sodium vapour with respect to each of the D lines.

Rubens has found that when the values of $\mu_\infty^2$, $C_1$, $C_2$, $\lambda_1$, and $\lambda_2$ have been determined, Sellmeier's formula gives correct values for the refractive indices of rock-salt, sylvine, fluor-spar, and quartz, over the entire range of wave-lengths to which these substances are transparent (Fig. 188). For instance, he found that the refractive index of sylvine is represented by the equation—

$$\mu^2 = 4·5531 + \frac{0·0150}{\lambda^2 - 0·0234} + \frac{10,747}{\lambda^2 - 4517·1}.$$

Here the wave-lengths are measured in microns. As already explained, there will be absorption bands for the wave-lengths which satisfy the equations—

$$\lambda^2 = 4517·1 \text{ and } \lambda^2 = 0·0234.$$

Corresponding to $\lambda = \sqrt{4517·1} = 67·2$ microns, we have an absorption band in the extreme infra-red part of the spectrum. Corresponding to $\lambda = \sqrt{0·0234} = 0·153$ microns (or 1530 tenth-metres), we have an absorption band in the ultra-violet. Thus, for intermediate values of $\lambda$, the refractive index can be represented by a curve like that for $\mu$ between the two absorption bands in Fig. 201.

**Motions of the Vibrating Particles.**—Let light-waves of period T be transmitted through a medium containing particles with free vibration periods equal to $T_1$. Then if $a$ is the amplitude of the light-waves, while $\alpha$ is the amplitude of the S.H.M., executed by the vibrating particles, we have (p. 256)—

$$\alpha = a\frac{T^2}{T^2 - T_1^2}.$$

When $T = T_1$, we have $a = \infty$. Thus, for waves of periods nearly equal to those of the vibrating particles, the amplitudes of the vibrations executed by the latter will be very great. The vibrating particles, being set in violent motion, absorb a considerable amount of energy. It appears that in some cases the vibrating particles are actually torn away from the atoms to which they belong. Thus, when ultra-violet light falls on an insulated metallic body charged with negative electricity, the charge is quickly dissipated: it is found that particles possessing masses equal to about 1/1000 of the mass of a hydrogen atom are given off, and these carry the negative charge away. Ultra-violet light seems to exercise a similar dissociating action on ordinary air.

The effect produced on the molecules of the absorbing body has been explained as follows by Lord Kelvin:—

"I believe that the first effect when light begins, of period exactly equal to $T_1$, is that each sequence of waves throws some energy into the molecule. That goes on until somehow or other the molecule gets uneasy. It takes in (owing to its great density relative to the ether) an enormous quantity of energy before it gets particularly uneasy. It then moves about, and begins to collide with its neighbours, perhaps, and will therefore give you heat in the gas if it be a gaseous molecule. It goes on colliding with other molecules, and in that way imparting its energy to them. This energy is carried away (as heat) by convection, perhaps. Each molecule set vibrating in that way becomes a source of light, and we may thus explain the radiation of heat from the molecule after it has been got into it by sequences of waves of light."

When light of a certain wave-length is absorbed, the disturbance of the molecules may set in motion vibrating particles, of periods greater than that of the incident waves. In this case we have fluorescence. If the particles continue vibrating for some time after the light has ceased to be incident, we have phosphorescence. In other cases the molecular disturbance may produce definite chemical changes, as when light is incident on silver chloride, or a mixture of hydrogen and chlorine.

**Selective Reflection.**—The effective density, $\rho'$, of a medium

possessing only one set of vibrating particles, for waves of period T, will be given by the equation (p. 283)—

$$\frac{\rho'}{\rho} = 1 + nK\frac{T^2}{T^2 - T_1^2},$$

where $\rho$ is the true density of the ether. It follows that the medium will possess an infinitely great density for waves of period equal to $T_1$, and the consequence will be that waves of this period will be strongly reflected from the surface of the medium. Thus we see that **a medium will strongly reflect waves of lengths approximately equal to those which it absorbs.**

EXPT. 59.—Obtain a small piece of plate glass, and suspend this by a piece of thin copper wire bound round its edges. Dip the piece of glass vertically into a beaker containing a strong solution of fuchsine in absolute alcohol, at about 30° to 40° C. On removing the glass from the solution, the alcohol quickly dries off, leaving a beautiful polished surface layer of fuchsine.

It will be noticed that the light reflected from the surface layer is of a yellowish colour. Fuchsine absorbs yellow and green light. The light reflected from the layer, and that transmitted through it, should be examined spectroscopically.

As already pointed out, the values of the constants in Sellmeier's formula determined by Rubens for sylvine, indicate that that substance possesses an absorption band corresponding to $\lambda = 67{\cdot}2$ microns. Consequently, waves of this length should be much more strongly reflected from sylvine than those of smaller lengths, and by repeated reflections from plane sylvine surfaces the *residual rays* (as Rubens has called them) should comprise only wave-lengths in the immediate vicinity of 67·2 microns. On trying this experiment, Rubens found that the residual rays for sylvine had a wave-length equal to 61·1 microns. These constitute the longest wave-lengths yet dealt with. In a similar manner he found that the residual rays for rock-salt and fluor-spar possess wave-lengths in satisfactory agreement with those calculated from Sellmeier's formula.

**Metallic Reflection.**—The strong reflection of light which occurs at a polished metallic surface is probably due to selective reflection, as explained above. A thin layer of silver on glass, which is almost entirely opaque to the longer waves in white

light, will yet be found to be fairly transparent to the violet and ultra-violet rays; the faint violet light of the electric arc can be seen through it, although the brilliant crater is rendered extremely dim. This proves that the reflection of silver is selective. Similarly, thin gold leaf, which reflects yellow light, transmits greenish-blue light.

**Metallic Refraction.**—Kundt has examined the refraction of light by very small acute-angled metallic prisms. He found that in the case of silver, gold, and copper, the light is deviated toward the refracting edge of the prism. The following values for the refractive indices of metals were obtained from the deviations and angles of the prisms in the manner explained on p. 90.

| Metal. | $\mu$ | Metal. | $\mu$ |
|---|---|---|---|
| Sodium . . . . . | 0·12 | Platinum . . . . | 1·64 |
| Silver . . . . . . | 0·27 | Iron . . . . . | 1·73 |
| Gold . . . . . . | 0·58 | Nickel . . . . . | 2·01 |
| Copper . . . . . | 0·65 | Bismuth . . . . . | 2·26 |

Kundt has pointed out that the refractive indices are approximately in the same proportions as the specific electrical resistances of the metals.

The refractive index of metallic sodium, given above, was determined by Drude. It is the smallest refractive index known. The velocity of light in sodium is about ten times as great as *in vacuo*. Prof. R. W. Wood has found that, for $\lambda = 5000$, the refractive index of amorphous selenium is equal to 3·13.

**Refractive Equivalents.**—By enclosing a gas in a hollow glass prism, its refractive index for various pressures may be measured. It has been found that the refractive index, $\mu$, of a transparent gas is in general only slightly greater than unity. Further, **Gladstone and Dale** established the law that **$(\mu - 1)$ is proportional to the density of the gas**, or, if $d$ is the density of the gas, $(\mu - 1)/d$ is constant for the gas, whatever may be its pressure.

In the case of a single gas, the number of vibrating particles per unit volume will be proportional to the number of gas molecules per unit volume. Thus, in equation (2) (p. 370), since $n_1$ and $n_2$ will both be

proportional to the number of gas molecules per unit volume, or to the density of the gas, we shall have, for light of a given wavelength, $\lambda$—

$$(\mu^2 - 1) \propto d, \text{ or } \frac{\mu^2 - 1}{d} = \text{a constant.}$$

But $(\mu^2 - 1) = (\mu + 1)(\mu - 1)$, and $\mu$ is very nearly equal to 1; thus $(\mu + 1)$ will be sensibly equal to 2, and we shall have $(\mu - 1) \propto d$.

**Lorentz and Lorenz** have established, on theoretical grounds, the law that—

$$\frac{\mu^2 - 1}{\mu^2 + 2} \propto d, \text{ or } \frac{\mu^2 - 1}{\mu^2 + 2} \cdot \frac{1}{d} = \text{a constant.}$$

When $\mu$ is nearly equal to 1, this relation degenerates into Gladstone and Dale's law. The formula of Lorentz and Lorenz is, however, much more general, since it can be used to calculate the refractive index of a vapour from that of the corresponding liquid, or *vice versâ*.

*Example.*—Hydrogen gas at 0° C. and 760 mm. pressure has a density of 0·0000896 gram per c.c., and its refractive index is equal to 1·000138. According to Dewar, liquid hydrogen has a density equal to 0·068 gram per c.c. Calculate the refractive index of liquid hydrogen.

For hydrogen gas—

$$\frac{\mu_1^2 - 1}{\mu_1^2 + 2} = \frac{(1\cdot000138)^2 - 1}{(1\cdot000138)^2 + 2} = \frac{\cdot000276}{3} \text{ nearly.}$$

If $\mu$ is the refractive index of liquid hydrogen, by the law of Lorentz and Lorenz—

$$\frac{\mu^2 - 1}{\mu^2 + 2} \cdot \frac{1}{0\cdot068} = \frac{0\cdot000276}{3 \times 0\cdot0000896} = \frac{276}{269}.$$

$$\therefore \frac{\mu^2 - 1}{\mu^2 + 2} = \frac{0\cdot068 \times 276}{269} = 0\cdot070.$$

$$\mu^2(1 - 0\cdot070) = 1 + (2 \times 0\cdot070) = 1\cdot14.$$

$$\therefore \mu^2 = \frac{1\cdot14}{\cdot93} = 1\cdot22,$$

$$\text{and } \mu = \sqrt{1\cdot22} = 1\cdot11.$$

Professor Dewar has, by experiment, found the refractive index of liquid hydrogen to be equal to 1·12.

**Polarisation by Tourmaline.**—When light is incident normally on a crystal of tourmaline, the waves, in which the displacements are parallel to a certain direction, are transmitted,

while those in which the displacements are perpendicular to this direction are almost completely absorbed. This can be explained by supposing that, tourmaline being a crystal, the molecules are regularly arranged, and the vibrating particles possess different periods according as they oscillate in the direction of the axis of the crystal, or perpendicular to that direction (p. 281). The period of vibration of the particles in one of these directions agrees sufficiently well with the period of light-waves for absorption to occur; waves in which the displacements are perpendicular to this direction are transmitted, since the period of the vibrations of the particles in the corresponding direction do not agree with the periods of the light-waves.

In support of this view of the polarisation of light by tourmaline, it may be mentioned that Kirchhoff and Stewart independently observed that the radiations emitted by a heated tourmaline plate are absorbed by another tourmaline plate if the axes of the two are parallel, but are transmitted by the second plate if the axes of the two are perpendicular. This shows that the radiations emitted by the heated tourmaline are similar to those which it absorbs, which is an instance of Kirchhoff's law (p. 339).

## Questions on Chapter XV

1. Describe and explain—

(1) The difference between the spectra produced by glowing solids and gases respectively.

(2) The effect of gradually increasing the thickness of a medium, a thin layer of which gives an absorption spectrum, consisting of several distinct narrow bands placed near to each other in the spectrum.

2. Describe the principal characteristics of the phenomenon of fluorescence, and describe experiments by which its relation to phosphorescence has been determined.

3. Describe a method of rendering the ultra-violet portion of the spectrum visible.

4. Describe Becquerel's phosphoroscope. What light do experiments made with this apparatus throw upon the relation between phosphorescence and fluorescence?

5. Describe a method of determining the time during which phosphorescence lasts after the exciting radiation is cut off, in the case of bodies for which this interval is small.

6. To what do you suppose dispersion is due? Illustrate your answer by reference to some mechanical system in which the velocity of wave propagation depends greatly on the period.

7. Give some account of the phenomenon of anomalous dispersion, and of the theory that accounts for it.

## Practical

1. You are supplied with magnesium wire, specimens of various substances, and a sheet of paper impregnated with calcium sulphide. Arrange the substances in the order of their absorption of the rays which excite phosphorescence.

# CHAPTER XVI

## INTERFERENCE

**General Principles.**—A general account of the effects produced by the interference of waves radiating from two similar sources has already been given (p. 317), and Fresnel's double mirror experiment has been described. In the present chapter some consideration will be given to other methods of producing optical interference phenomena, and closer attention will be directed toward the nature of the effects produced.

Let A, B, (Fig. 204) be two points from which similar light-waves radiate. The waves originating at A and B must possess equal periods, and, on starting, their phases must be either equal, or must differ by an amount which remains constant. In such cases, as we have seen, there will be a number of continuous regions in which the ether remains permanently stationary. Between any two consecutive stationary regions, there will be a region of maximum displacement. If a screen, DE, is placed in front of the sources A and B, its illumination will not be uniform; a number of bands, alternately bright and dark (if the light is monochromatic) or brilliantly coloured (if the light is white), will be seen.

Fig. 204.—Illustrates the Theory of Interference.

Join A, B, and bisect the line AB in F. Through F draw FC perpendicular to AB. Let the screen DE be perpendicular to

FC. Then, if the waves start from A and B in the same phase, the point C will be brilliantly illuminated, no matter what may be the length of the light-waves. For the distances between C and A, C and B, are equal, and therefore the waves from A and B will arrive at C in the same phase. C is the centre of the central interference fringe. This fringe will be white if white light is used.

Let us now determine the nature of the illumination at a point P, at a distance CP from C. Join PA, PB, and PF. With P as centre, and radius PA, describe the circular arc AG. If the distance AB is very small in comparison with PA, the line AG will be sensibly straight, and perpendicular to both PF and BG. BG will be equal to the **distance retardation** of the wave from B with respect to that from A. If BG is equal to half a wave-length, or any odd number of half wave-lengths, the point P will be dark; for in such cases the waves from B will arrive at P one half-period after those from A, or the two sets of waves on reaching P will differ in phase by $\pi$. If BG is equal to one wave-length or any whole number of wave-lengths, the point P will be brightly illuminated; for in such cases the two sets of waves will arrive at P in the same phase, and will therefore reinforce each other.

It is easily seen that the triangles BGA and PCF are similar. For these triangles possess right angles at G and C respectively; and since AG is perpendicular to PF, while AB is perpendicular to FC, the angle BAG is equal to the angle PFC. Hence, the angle ABG is equal to the angle FPC, and the triangles BGA and PCF are similar in all respects.

Let CP $= x$, while AB $= d$, and PF $=$ D. Since the distance CP is small, the line FC will be approximately equal to FP, or to D.

From the similarity of the triangles PCF and BGA, we have—

$$\frac{\text{CP}}{\text{FP}} = \frac{\text{BG}}{\text{AB}}; \quad \therefore \quad \frac{x}{\text{D}} = \frac{\text{BG}}{d}.$$

For P to be the centre of a dark band, BG must be equal to some odd number of half wave-lengths. Let $\lambda$ be the wave-length of the light; then BG $= (n + \frac{1}{2})\lambda$, where $n$ may have any integral value from zero upwards. Thus, **for P to be the centre of a dark band,**

$$x = \frac{\text{D}}{d}(n + \tfrac{1}{2})\lambda,$$

For P to be a point of maximum illumination, BG must be equal to any whole number of wave-lengths, say $n\lambda$, where $n$ may have any integral value from zero upwards. Thus, **for P to be the centre of a bright fringe**—

$$x = \frac{D}{d} n\lambda.$$

The distance between the centres of the $n$th and $(n+1)$th bright fringes is equal to—

$$\frac{D}{d}\left\{(n+1)\lambda - n\lambda\right\} = \frac{D}{d}\lambda.$$

Thus the bright fringes will be equidistant from each other.

Since $\lambda$ is very small, it becomes apparent that, for the breadth of a band to be of appreciable magnitude, the ratio $D/d$ must be very large; *i.e.* the distance between the wave sources must be very small in comparison with their mean distance from the screen.

It also follows that the breadth of an interference fringe is directly proportional to the wave-length of the light employed. Thus, since red light produces wider bands than blue light, the wave-length of red light is greater than that of blue light (p. 323). If D and $d$ are known, the measurement of the distance between two consecutive bright fringes gives us the means of determining the wave-length of the light employed. It may here be remarked that all determinations of wave-lengths depend, either directly or indirectly, on some form of interference experiment.

Special precautions must be taken to ensure that the waves emitted at the two sources are either in the same phase, or differ in phase by some constant amount. In Fresnel's double mirror experiment two virtual images of an illuminated slit were used as wave sources; and, in general, one source must be the image of the other, or both must be images of some other source, in order to ensure a constant relation between the phases of the emitted waves. Further, the sources must be either approximate points, or lines, of light. Sources of finite magnitude could be decomposed into a number of corresponding linear or point elements, and a separate series of interference fringes would be formed by the waves from each pair of corresponding

elements; superposition of the various sets of fringes would produce indistinctness, or even uniform illumination.

We thus see that no interference phenomena could be produced by using two separate candle-flames, any more than two brass bands playing different tunes in the same street could produce silence. Each point of a candle-flame is a wave source, and there is no constant relation between the phases of the waves emitted by any two points of the same flame, or of different flames.

FIG. 205.—Illustrates the Displacement of Interference Fringes.

**Displacement of the Fringes.**—Let us now suppose that light from one of the sources (say A, Fig. 205) has to traverse a thin lamina, L, of a transparent substance, before reaching the screen, while light from the other source, B, reaches the screen directly. Let $t$ be the thickness of the lamina, and $\mu$ its refractive index. If $V_0$ is the velocity of light in air, while V is the velocity of light in the lamina, then $V_0/V = \mu$. Before a light-wave from A can reach P, it must travel a distance $(AP - t)$ in air, and a distance $t$ in passing through the lamina. The time required for this journey is equal to—

$$\frac{AP - t}{V_0} + \frac{t}{V}.$$

The time required for a light-wave to reach P from B is equal to $BP/V_0 = (BG+GP)/V_0 = (BG+AP)/V_0$. Hence, the **time retardation** of the waves from B behind those from A is equal to—

$$\frac{BG}{V_0} + \frac{AP}{V_0} - \left\{\frac{AP - t}{V_0} + \frac{t}{V}\right\} = \frac{BG}{V_0} + t\left(\frac{1}{V_0} - \frac{1}{V}\right).$$

The point P will be the centre of the $n$th bright fringe if this time retardation amounts to $n$T, where T is the period of the waves. Thus, **for P to be the centre of the $n$th bright fringe**—

$$\frac{BG}{V_0} + t\left(\frac{1}{V_0} - \frac{1}{V}\right) = nT\,;\ \therefore\ BG = nV_0T + t\left(\frac{V_0}{V} - 1\right).$$

Remembering that $BG = d\frac{x}{D}$, as already determined, and $V_0T = \lambda$, we have—

$$d\frac{x}{D} = n\lambda + (\mu - 1)t. \quad \therefore \quad x = \frac{D}{d}(n\lambda + (\mu - 1)t). \qquad (1)$$

This gives us the distance $x$ = CP of the $n$th bright fringe from the point C at which the central bright fringe would be formed if the lamina L were absent. Owing to the presence of the lamina L, the central fringe will be displaced to a distance $x_0$ determined by substituting $n = 0$ in (1); in **all cases the central fringe is the one at which the waves from A and B arrive after journeys of equal duration.** Thus, the displacement of the central fringe, due to the presence of the lamina L, is given by—

$$x_0 = \frac{D}{d}(\mu - 1)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, if V is less than $V_0$ (*i.e.* if light travels more slowly in a refracting medium than in air), $\mu$ will be greater than unity, and the value of $x_0$ will be positive. In this case the central fringe will be displaced toward P (Fig. 205). If the contrary were the case, $x_0$ would be negative, and the displacement of the central fringe would be in the direction of P′. Experiment shows that the central fringe is displaced toward the side on which the lamina is situated, *i.e.* toward P. Thus, we obtain additional evidence that light is transmitted more slowly in a refracting medium than in air.

The first bright fringe will occur at a distance $x_1$ from C, given by—

$$x_1 = \frac{D}{d}(\lambda + (\mu - 1)t).$$

Hence, the distance $\beta$ between the first and the central fringe is given by—

$$\beta = \frac{D}{d}(\lambda + (\mu - 1)t) - \frac{D}{d}\cdot(\mu - 1)t = \frac{D}{d}\lambda,$$

which is the result already obtained for the breadth of an interference fringe.

Substituting $D/d = \beta/\lambda$ in (2), we obtain—

$$x_0 = \frac{\beta}{\lambda}(\mu - 1)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the thickness, $t$, of the lamina is known, and $\beta$ and $x_0$ are measured, we can determine the refractive index, $\mu$, of the lamina for light of wave-length $\lambda$. If, on the other hand, $\mu$ is known, we can in a similar manner determine the thickness, $t$, of the lamina.

PROBLEM.—Fresnel's fringes are produced with homogeneous light of wave-length $6 \times 10^{-5}$ cms. A thin film of glass (refractive index 1·5) is introduced into one of the interfering rays, upon which the central bright band shifts to the position previously occupied by the fifth bright band from the centre (not counting the central band itself). The ray traverses the film perpendicularly. What is the thickness of the glass film? (Board of Education, Honours, 1894.)

Here, $x_0 = 5\beta$. Substituting in (3) we obtain—

$$5 = \frac{\cdot 5}{6 \times 10^{-5}}t. \quad \therefore \; t = \frac{5}{\cdot 5} \times 6 \times 10^{-5} = 6 \times 10^{-4} \text{ cms.}$$

It should be noticed that the displacement of the central fringe cannot be determined when monochromatic light only is used, since in that case all of the fringes are precisely similar, and there is no means of distinguishing the central from any other fringe. When white light is used, the central fringe is white, while the rest of the fringes are coloured, and are distributed symmetrically on either side of the central fringe. The usual procedure is to determine the approximate position of the central fringe, using white light as an illuminant, and then, using monochromatic light (conveniently obtained by the use of a mercury vacuum lamp, p. 333), to determine the exact position of the fringe for the particular wave-length of light employed.

**Fresnel's Bi-prism.**—Fresnel invented a very simple means of obtaining interference fringes by the use of a *bi-prism*, *i.e.* two acute-angled prisms placed base to base. In practice both prisms are ground from the same piece of glass. Light from a slit perpendicular to the plane of the paper at O (Fig. 206) falls on the bi-prism CED, and two virtual images, A and B, are formed by refraction. The light from these images produces

interference in the manner already described. Since the fringes are narrow, it is usual to observe them by the aid of a lens or eye-piece. Let a lens or eye-piece be placed to the right of M

FIG. 206.—Production of Interference Fringes by the aid of a Bi-prism.

(Fig. 206), so that its principal focus is at M, while the screen FG is removed; then on looking through the lens one sees a magnified image of the fringes that would have been formed on the screen. In other words, the eye, aided by the lens, sees the interference phenomena produced in the focal plane of the lens. Thus the only difference produced by moving the lens to the right is that the fringes will appear wider (compare Fig. 175, p. 317).

FIG. 207.—Photograph of Bi-prism Interference Fringes.

I am indebted to Prof. Chant for the photograph of the bi-prism fringes, of which Fig. 207 is a reproduction. The true interference fringes are seen in the centre of the figure. The wider external bands are due to diffraction, as will be explained in Chap. XVII.

EXPT. 60. *To determine the wave-length of light from a sodium flame, by the use of a bi-prism.*—Warm a small sheet of plate glass, coat this with paraffin wax, and then apply a thin sheet of tinfoil. After smoothing the latter out, allow the glass to cool, and then cut a narrow

slit in the tinfoil by the aid of a sharp knife or a razor. Mount the plate, with the slit vertical, near one end of a long bench (S, Fig. 208). Mount a bi-prism, B, with its refracting edges vertical, at a distance of 9 or 10 inches from the slit. A low-power travelling microscope,[1] M, placed at a distance of 1 or 2 feet from B, may be used to view the fringes. The only adjustment required is to bring the edge, E, of the bi-prism exactly parallel to the slit. A luminous gas-flame, G, may be used as

FIG. 208.—Method of observing Bi-prism Fringes.

an illuminant during adjustment; subsequently a Bunsen flame, which passes through an iron wire ring coated with common salt (p. 333), must be used. Looking through the microscope, the interference phenomena produced at F, in the focal plane of the microscope, will be seen. The distance from F to the slit S will give the value of D (p. 391). The position of F may be found by moving a needle about in front of the objective of the microscope, until it is distinctly seen without parallax; it will then be at F. On moving the travelling microscope laterally, parallel to itself, one after another of the bright fringes will be focussed on the intersection of the cross-wires. Obtain readings for the positions of two fringes separated by an observed number of dark bands; the distance between these fringes, divided by the number of intervening dark bands, will give $\beta$ (p. 391). To find $d$, the distance between the virtual images which act as wave sources, without altering the adjustments already made, place a lens, L, between S and F, in such a position that the two images of the slit illuminated by the sodium flame are seen in focus through the microscope. Real images of the wave sources will now be formed at F, and the distance between them must be measured by bringing the cross-wire into coincidence with one after the other. If $\delta$ is the distance through which the microscope is moved parallel to itself, it is easily seen (p. 72) that—

$$d = \frac{SL}{LF}\delta.$$

Finally (p. 391),

$$\lambda = \frac{d}{D}\beta.$$

[1] A travelling microscope, suitable for this and many other similar experiments is made by Mr. W. Wilson, 1 Belmont Street, Chalk Farm, N W.

The bi-prism fringes possess a peculiarity by which they are distinguished from those produced by other methods. In Fresnel's double mirror experiment, the breadth of a fringe is directly proportional to the wave-length of the light employed. In the case of the bi-prism fringes this relation does not hold. Since the blue rays are deviated by the bi-prism to a greater extent than the red rays, the images A and B (Fig. 206) will be narrow spectra when white light is employed. The images of the source O, due to the blue rays, will be more widely separated than those due to the red rays, and thus the interference fringes produced by blue light will be narrower, in comparison with those produced by red light, than would be the case in the double mirror experiment. A great deal of light is transmitted by this bi-prism, so that the fringes are very bright.

**Lloyd's Single Mirror Fringes.**—Dr. Lloyd obtained fringes by causing light from a narrow slit to be split up into two

B

pencils, one direct, and the other reflected from a polished black glass surface. In this case the wave sources are the illuminated slit itself (B, Fig. 209), and the virtual image, A, of the slit obtained by reflection at the glass surface. The slit must be adjusted to be parallel to the reflecting surface. As will be seen from Fig. 209, the point M, in which the screen is cut by the perpendicular through the centre of the line AB (compare Fig. 204), will lie in the plane of the reflecting surface. Thus, the point M is not illuminated by the reflected light, and the central fringe consequently is not formed; it can, however, be brought into view by placing a thin film of glass or mica in the path of the directly-transmitted pencil. Owing to the retardation of the waves which traverse the film, the position of the central fringe is displaced toward P (p. 393). If the film is of suitable thickness, the position of the central fringe will be

situated on that part of the screen illuminated by the reflected waves. A similar result may be obtained, when the screen is removed and the fringes are directly observed by the aid of a lens, by tilting the lens in a vertical plane.

Dr. Lloyd observed that the central fringe is black, instead of being white as in most other cases. On either side of the central black band is a white fringe, the rest of the fringes being coloured. This indicates that, **when light is reflected from an optically denser medium, the phase of the reflected waves differs by $\pi$ from that of the incident waves** (p. 283). The central fringe is reached by waves from A and B (Fig. 209) in equal times; but the reflected waves virtually start from A, differing in phase by $\pi$ from those starting from B, so that the two sets of waves annul each other over the central band. At neighbouring points, at which the waves from A arrive half a period earlier or later than those from B, there will be bright fringes. The approximate agreement of the bright fringes of various colours renders the first fringes on either side of the central black band approximately white.

FIG. 210.—Interference of Ripples on the Surface of Mercury. (From a photograph by Dr. Vincent.)

The method of production of Lloyd's fringes may be made clearer by the aid of Fig. 210. This is a reproduction of an instantaneous photograph, due to Dr. Vincent, of ripples on the surface of mercury. A single glass style, attached to the prong of a vibrating tuning-fork, generates

circular ripples which are reflected from a small set-square lying on the mercury surface. On those parts of the mercury surface where the direct and reflected ripples overlap, stationary regions are formed by interference.

**Achromatic Interference Fringes.**—In all of the methods previously discussed, only a limited number of fringes are visible when white light is used as an illuminant. This is due to the circumstance that the distance between two consecutive bright fringes is different for different wave-lengths of light. The central fringes for all wave-lengths will be superposed; but, as we pass in any direction from the central fringe, there will be greater and greater confusion, due to the overlapping of fringes of various colours, some of which are dark, while others are bright. If an arrangement can be devised which will render the distance between two consecutive bright fringes the same for all wave-lengths, then, by using white light as an illuminant, a system of fringes will be obtained which are alternately black and white. Such fringes are said to be **achromatic**. The system of fringes corresponding to any wave-length of light will now be exactly similar (except in colour) to the system for any other wave-length, and the bright fringes corresponding to all parts of the spectrum will be superposed, leaving black intermediate spaces where the dark fringes are superposed.

Inspection of the formula for $\beta$, the distance between the centres of any two neighbouring bright fringes, gives us a clue to a method by which achromatic fringes may be obtained. We have (p. 393)—

$$\beta = \frac{D}{d}\lambda.$$

If we can secure that the ratio $\lambda/d$ shall remain constant for all values of $\lambda$, then $\beta$ will remain constant. Fig. 211 represents an arrangement which secures the approximate constancy of $\lambda/d$. Light from a horizontal slit, S, rendered parallel by a convergent lens, $L_1$, is refracted by a prism, P, and then brought to a focus by the convergent lens $L_2$. A pure spectrum, RV, is thus formed, and by reflection from a plate of black glass, G, a virtual image, V′R′, of this spectrum is likewise produced. If the slit S were illuminated by red light, we should obtain two horizontal linear images, R and R′; if by blue light, two hori-

zontal linear images, V and V′, would be produced. Light from either pair of images will produce interference. For red light, $d = \mathrm{RR'}$, while for blue light, $d = \mathrm{VV'}$. Let $\lambda_1$ and $\lambda_2$ be the respective wave-lengths of the red and blue rays. Then, if we arrange that—

$$\frac{\mathrm{RR'}}{\mathrm{VV'}} = \frac{\lambda_1}{\lambda_2}$$

the red and blue fringes will be equal in breadth, and will therefore be superposed without confusion. Fringes produced by light from other parts of the spectra will also be approximately superposed, so that the resultant fringes will be achromatic in the same sense that an ordinary telescope is. When

FIG. 211.—Method of producing Achromatic Interference Fringes.

the adjustments are properly made, many hundreds of fine bands, alternately white and black, are brought into view. Colour only appears near the edges of the field.

As will be explained in a succeeding chapter, a diffraction grating can be used to form a spectrum, in which the dispersion is accurately proportional to the wave-length. If the spectrum RV (Fig. 211) is produced by the aid of a diffraction grating instead of a prism, then the superposition of the bands for any two wave-lengths will secure the accurate superposition of the bands for all parts of the spectrum. By this means true achromatic bands may be obtained.

**The Divided Lens Method.**—M. Billet invented a convenient method of producing interference by the aid of a convex lens cut into halves. These are mounted so that the distance between them can be adjusted at pleasure, and light from a slit, S (Fig. 212), is allowed to fall on the combination. When the halves of the lens are separated by a short distance, two real

images of the slit will be formed, at B and A respectively. Light from these images interferes as in the similar cases already discussed.

This apparatus may be conveniently used to investigate the interference of polarised light. In the first place, a plate of tourmaline, T, may be placed in front of the slit S: interference still takes place as

FIG. 212.—Production of Interference Fringes by the aid of a Divided Lens.

before. If the tourmaline, T, is removed, and two tourmaline crystals exactly equal in thickness are placed at B and A, then it is found that interference still occurs when the axes of the crystals are parallel; in this case the vibrations in both transmitted pencils are in the same direction. When the axes of the crystals are placed at right angles to each other, all traces of interference vanish. In this case the directions of vibration in the transmitted pencils are at right angles to each other, so that interference cannot be produced. This experiment gives us additional evidence as to the transverse direction of the displacement in light-waves.

**Colours of Thin Films.**—When a very thin film of a transparent substance is exposed to light from an extended source (such as a luminous gas-flame, or a bright sky), brilliant colours are seen. The most familiar instances of this coloration are afforded by soap-bubbles and the thin films produced by allowing oil to spread out over the surface of water. The production of colour in these circumstances can be consistently explained in terms of the interference of light-waves. Newton studied the colours of thin films, and explained them on his "theory of fits" in conjunction with the corpuscular theory of light (p. 235). In view of the conclusive disproof of the corpuscular theory, the interferential explanation alone has interest for us at the present time.

In what follows it must be borne in mind that a ray of light is the path of a disturbance from a particular point in a light-wave. When we

speak of the interference of rays of light, it must be understood that the interference really takes place between the waves travelling along the rays.

**Colours by Reflected Light.**—Let AB (Fig. 213) represent a light-ray incident at B on one of the parallel bounding surfaces of a thin transparent lamina. The incident ray is split up into a reflected ray, BC, and a transmitted ray, BD. The transmitted ray is partly reflected at D, and thus gives rise to the ray DE, which in its turn gives rise to the ray EF refracted into the air. Since the bounding surfaces of the film are parallel, the rays BC and EF will be parallel; if the film is very thin, the rays will also be so close to each other that a single ray will be formed by their combination.

FIG. 213.—Rays Reflected and Transmitted by a Thin Film.

The waves reflected from B, a point on the surface of an optically denser medium (p. 283), will differ in phase by $\pi$ from the incident waves. The ray BDEF experiences no sudden phase change, but in travelling over the path BDE it is retarded behind the ray directly reflected at B. If the phases of the wave disturbances forming the rays BC and EF differ by $\pi$, or any odd multiple of $\pi$, interference will occur; if the phases differ by any even multiple of $\pi$, the two rays will reinforce each other.

Let us now suppose that the thickness of the film is small in comparison with the wave-length of the light employed as an illuminant. In that case, since the distance BDE will be small in comparison with the wave-length of light, the ray EF will experience no appreciable retardation, and its phase will be sensibly equal to that of the incident ray. The phase of the reflected ray, BC, differs by $\pi$ from that of the incident ray. Consequently, the rays EF and BC differ in phase by $\pi$. If the amplitudes of the corresponding waves are equal, the two rays BC and EF will together produce darkness.

A similar result will follow if the film is optically less dense than the media above and below it. In that case the ray reflected at B will

experience no phase change, but the ray internally reflected at D will experience a phase change of $\pi$, since in this case reflection occurs at a denser medium.

When the film is exceedingly thin, the rays BC and EF will annul each other, whatever may be the length of the corresponding waves. Thus, **a film, of which the thickness is small in comparison with the wave-length of light, will appear to be black by reflected light.**

Let us now suppose that monochromatic light is used as an illuminant, and that the thickness of the film can be gradually increased. For a certain thickness the retardation due to the path BDE will amount to half a wave-length; in this case the rays EF and BC will both differ in phase by $\pi$ from the incident ray AB, so that their phases will be equal. The rays BC and EF will thus reinforce each other, and the film will appear brightly illuminated.

As the thickness of the film increases, the retardation along the path BDE will successively amount to 2, 3, 4, . . . half wave-lengths. When the thickness is such that the retardation along the path BDE amounts to an even number of half wave-lengths, the film will appear black by reflected light; in the other cases the film will appear brightly illuminated.

If AB is a ray of blue light, the thickness of the film, in order that a retardation of half a wave-length shall occur along the path BDE, will be less than in the case of red light, since the wave-length is less for blue than for red light.

Let us now suppose that we have a film which increases in thickness uniformly from one edge to the other. Let the thickness at the thin edge be small in comparison with the wave-length of violet light. Then, if the film is illuminated by monochromatic light, a number of bright bands separated from each other by dark intervals will be seen. The distance between two bright bands will be less for blue than for red light. If the film is illuminated by white light, the thin edge of the film will appear black. Passing along the film in the direction in which its thickness increases, we shall first reach a point where blue light is reflected, then points where green, yellow, and red lights are reflected. Passing still further in the same direction, we shall encounter coloured bands formed by the overlapping of bands of different breadths due to the different wave-lengths of light.

Ultimately we shall reach a point where the overlapping produces uniform illumination.

EXPT. 61.—Obtain a circular wire ring of about 2 inches diameter, soldered to a straight wire to act as support. Dip this ring in a prepared soap solution,[1] the surface of which has been freed from bubbles. On removal, a plane film of soap solution will be found to stretch across the ring. Insert the supporting wire in a hole in a wooden block, so that the plane of the ring is vertical, and cover the whole with a glass shade or inverted beaker. After a time brilliant bands of colour will be seen to follow each other down the film, until at last the upper part of it, when viewed by reflected light, appears perfectly black. The film is thinnest near its upper edge, and gradually thickens toward its lower extremity.

The colours of the bands, seen on a thin film when viewed by reflected white light, were classified by Newton as follows, starting from the black film of infinitesimal thickness :—

1. Black, blue, white, yellow, red.
2. Violet, blue, green, yellow, red.
3. Purple, blue, green, yellow, red.
4. Green, red.
5. Greenish-blue, red.
6. Greenish-blue, pale red.
7. Greenish-blue, reddish-white.

EXPT. 62.—View the soap film used in Expt. 61 through a piece of red glass. Notice the alternate bright and dark bands. Next view it through a piece of blue glass. Notice that the blue bands are narrower than the red ones were.

**Colours by Transmitted Light.**—The ray BD (Fig. 213) is partly reflected along DE, and partly transmitted into the medium above the film, thus giving rise to the ray DG. The reflected ray DE is partly reflected at E, giving rise to the ray EH, which in its turn gives rise to the transmitted ray HK. The latter is parallel to, and lies immediately beside, the ray DG. The ray DG is formed from the ray BD without any phase change. The ray HK is formed without any sudden phase change, if the film is optically denser than the media above and below it. If

[1] The following soap solution is recommended by Professors Reinold and Rücker. Fill a clean stoppered bottle three-quarters full of distilled water. Add oleate of soda amounting to one-fortieth part of the weight of water. Shake, and leave for a day, when the oleate of soda will be dissolved. Nearly fill the bottle with Price's pure glycerine, and well shake Leave the stoppered bottle for a week in a dark place, and then siphon off the clear liquid into another stoppered bottle, the scum being left behind. Add one or two drops of strong ammonia to each pint of solution, and keep in a dark place. The solution must not be warmed or filtered.

the film is optically less dense than the media above and below it, phase changes, each amounting to $\pi$, will occur on reflection at D and E, so that the total phase change due to this cause amounts to $2\pi$, which is equivalent to no phase change at all. Thus, in either case, the only difference in phase between the rays HK and DG will be due to the retardation produced along the path DEH. When the film is infinitesimally thin, the phases of the rays HK and DG will be equal, and these two rays will reinforce each other, so that the film will appear bright by transmitted light. In this case the whole of the incident light is transmitted, the reflected rays BC and EF interfering with each other. As already remarked, light can never be *destroyed* by interference (p. 318); if darkness is produced at a certain point, enhanced illumination must be produced elsewhere.

If we now imagine that the film increases in thickness, and monochromatic light is used as an illuminant, the rays DG and HK will interfere when the retardation along the path DEH amounts to any odd number of half wave-lengths. The retardation along the path DEH is obviously equal to that along the path BDE, and a retardation along BDE amounting to an odd number of half wave-lengths brings the rays BC and EF into the same phase, so that they reinforce each other. Thus it follows that, **when the film is of such a thickness that it appears bright by reflected light, it will appear dark by transmitted light,** and *vice versâ*.

When white light is used as an illuminant, the colour of the film, either by transmitted or reflected light, will be due to the suppression of certain wave-lengths by interference. Since in any case the wave-lengths which are suppressed in the reflected ray are present in the transmitted ray, it follows that **the colours of a film, when viewed by reflected and transmitted light respectively, will be complementary.**

EXPT. 63.—Confirm the above conclusions, by viewing a soap film by reflected and by transmitted light, first through a red glass, and then without the latter.

**Calculation of the Retardation.**—I. REFLECTED LIGHT.—As already explained, the incident ray AB (Fig. 214) gives rise to the directly reflected ray BC, together with the ray EF which has traversed the path BDE. Produce ED to L, making

DL = DB. Join LB. Then the line LB will be perpendicular to the parallel bounding surfaces of the film, and if the thickness of the film is equal to $\delta$, then LB = $2\delta$. Draw EP perpendicular to BC. Then EP will be parallel to the wave-fronts in the rays BC and EF.

FIG. 214.—Relative Retardation in Transmitted and Reflected Rays.

We may consider that the directly reflected wave starts from B to travel along the direction BC in air, at the same instant that the internally reflected wave starts from the point L to travel along LM in the substance of which the film is composed. We must now determine the time interval which elapses between the passage of the two waves through the plane EP perpendicular to the paper.

Let $V_0$ be the velocity of light in air. Then the directly reflected wave passes through the plane EP after an interval equal to $BP/V_0$ seconds, measured from the instant at which it started from B.

The internally reflected wave passes through the plane EP after an interval equal to LE/V seconds, measured from the instant at which it started from L; V being the velocity of light in the film. Thus, the time retardation of the internally reflected wave behind that directly reflected at B, is equal to $(LE/V) - (BP/V_0)$.

Draw BM perpendicular to ED. Then, since the opposite faces of the film are parallel, ED and BD are equally inclined to the face EB, and the angle MBE is numerically equal to $r$, the angle of refraction of the ray BD into the film. Thus, ME/EB = sin $r$. Similarly, the angle BEP is numerically equal to $i$, the angle of incidence of the ray AB; and BP/EB = sin $i$. Therefore, since $\sin i/\sin r = \mu$, we have—

$$BP = \mu \, ME.$$

The distance through which the directly reflected wave has travelled in advance of the plane EP, at the instant when the internally reflected wave reaches that plane, will obviously be equal to the time retardation of the two waves, multiplied by $V_0$; that is, to $LE \, . \, V_0/V - BP$, or $\mu \, . \, LE - BP$. Thus, the **distance retardation** between the waves is equal to—

$$\mu(LM + ME) - BP = \mu \, . \, LM.$$

But $LM = LB \cos BLM = 2\delta \cos r$. (BD makes equal

angles with the opposite faces of the film, and thus $\angle$ BDM = $2r$. Also, since the triangle BDL is isosceles, and BDM is its external angle, $\angle$ BDM = $\angle$ BLD + $\angle$ LBD = 2 . $\angle$ BLD). Hence, **the distance retardation of the internally reflected ray,** EF, **behind the directly reflected ray,** BC, **is equal to** $2\mu\delta \cos r$.

Taking account of the phase change of $\pi$ which occurs in the reflection at B, we see that, **for the rays** BC **and** EF **to interfere**—

$$2\mu\delta \cos r = n\lambda,$$

where $n$ is any positive integer (including 0), and $\lambda$ is the wave-length of the incident ray.

**For the rays** BC **and** EF **to reinforce each other**—

$$2\mu\delta \cos r = (n + \tfrac{1}{2})\lambda.$$

2. TRANSMITTED LIGHT.—The ray BD (Fig. 214) gives rise to the transmitted ray DG, together with the reflected ray DE. The ray DE is reflected at E, thus producing the ray EH, which in turn gives rise to the transmitted ray HK. Draw HS perpendicular to DG, and DR perpendicular to EH. Produce HE to T, making ET = ED. Then the internally reflected wave may be supposed to start from T at the instant when the transmitted wave starts from D. The distance in advance of the plane HS, covered by the transmitted wave, at the instant when the internally reflected wave passes through that plane, may be found as before, or more shortly as follows :—The path TH, in a medium of refractive index equal to $\mu$, is equivalent to a path equal to $\mu \times$ TH in air. Also, $\mu$ . RH = DS. Thus, the distance retardation of the internally reflected wave behind the transmitted wave is equal to—

$$\mu\text{TH} - \text{DS} = \mu(\text{TR} + \text{RH}) - \text{DS} = \mu\,.\,\text{TR} = 2\mu\delta \cos r$$

The only source of phase difference between the rays DG and HK is the retardation due to the difference in their paths. Thus, **for the rays** DG **and** HK **to interfere**—

$$2\mu\delta \cos r = (n + \tfrac{1}{2})\lambda,$$

where $n$ is any positive integer (including 0). **For the rays** DG **and** HK **to reinforce each other**—

$$2\mu\delta \cos r = n\lambda.$$

It thus becomes apparent that the film will assume different

colours as the angle at which it is viewed is varied. An increase in the value of $r$ produces the same effect as a decrease in $\delta$.

**Reason why an Extended Source of Light must be used.**—An eye placed near F or C (Fig. 213) will receive light, from the portion EB of the film, only along the rays BC and EF. (It must be remembered that these rays are supposed to be so close together that they practically coincide.) **These rays are both derived from the incident ray** AB, which proceeds from a particular point in the source.

When the rays BC and EF interfere, the portion EB of the surface will appear dark; otherwise it will appear to be brightly illuminated. Similar reasoning applies to the transmitted light. The eye will see different points on the film by means of rays primarily derived from different points in the source of light. Thus, an extended source of light is necessary for the perception of the general colour of the film. If an illuminated slit were used as a source of light, we should see only a coloured image of the slit reflected in the film, the film as a whole remaining dark.

**Newton's Rings.**—If two convex lenses are placed in contact with each other, a thin air film will be formed between them. Near the point of contact of the lenses, the thickness of the air film will be very small in comparison with the wave-length of light Consequently, the point of contact will be surrounded by a circular black spot, when viewed by reflected light, or by a bright spot, when viewed by transmitted light. The thickness of the air film increases as we proceed from the point of contact toward the periphery of the lenses. Since the surfaces of the lenses are portions of spheres, the thickness of the air film will be uniform for all points on a circle concentric with the point of contact. Thus, if monochromatic light is used to illuminate the lenses, the central black spot, seen by reflected light, will be surrounded by concentric bright circles separated by dark intervals. Each bright circle seen by reflected light will correspond to a dark circle seen by transmitted light, and *vice versâ*. When white light is used as an illuminant, the rings will be brightly coloured, the tint of a ring seen by reflected light being complementary to that of the corresponding ring seen by transmitted light. These coloured rings have been named after Newton, who carefully examined and described them.

Let Fig. 215 represent a lens of which the convex surface is in contact, at A, with the plane surface of a sheet of plate-glass. When viewed normally by reflected light, the points C and B, equidistant from A, will lie on a bright or dark circle, according as twice the distance BF is equal to an odd or even number of half wave-lengths of the incident light.

Fig. 215.—Illustrates the Production of Newton's Rings.

From A draw the diameter AK to the circle of which the curved section of the lens is an arc. Join CB, cutting AK in G. Let BF = GA = $\delta$. Let the diameter CB of the ring observed be equal to D. Then, if R is the radius of curvature of the lower surface of the lens—

$$(2R - \delta)\delta = \left(\frac{D}{2}\right)^2.$$

Since $\delta$ will be very small in comparison with R, we shall have—

$$\delta = \frac{D^2}{8R}.$$

For B and C to be situated on a bright ring—

$$2\delta = \frac{D^2}{4R} = (n + \tfrac{1}{2})\lambda = (2n + 1)\frac{\lambda}{2},$$

where $n$ has the values 0, 1, 2, 3, . . . &c., for the first, second, third, fourth, . . . &c., rings respectively. Thus, the diameters $D_1$, $D_2$, $D_3$, . . . &c., of the first, second, third, . . . &c., bright rings will have the values—

$$D_1 = 2\sqrt{R\frac{\lambda}{2}},\ D_2 = 2\sqrt{R\frac{3\lambda}{2}},\ D_3 = 2\sqrt{R\frac{5\lambda}{2}},\ \&c.$$

Thus, **the diameters of the bright rings will be proportional to the square roots of the odd natural numbers,** 1, 3, 5, 7, &c.

For B and C to be situated on a dark ring—

$$2\delta = \frac{D^2}{4R} = n\lambda.$$

When $n = 0$, we obtain $D = 0$, which corresponds to the centre of the black spot. The diameters $D_1'$, $D_2'$, $D_3'$, . . . &c., of the first, second, third, . . . &c., dark rings will be given by—

$$D_1' = 2\sqrt{R\lambda},\ D_2' = 2\sqrt{2R\lambda},\ D_3' = 2\sqrt{3R\lambda},\ \ldots\ \&c.$$

Thus, **the diameters of the dark rings are proportional to the square roots of the natural numbers** 1, 2, 3, 4, 5, . . . &c.

It may be left as an exercise for the student to prove that, by transmitted light, the diameters $D_1$, $D_2$, $D_3$, . . . &c., of the first, second, third, . . . &c., bright rings will have the values—

$$D_1 = 2\sqrt{R\lambda},\ D_2 = 2\sqrt{2R\lambda},\ D_3 = 2\sqrt{3R\lambda},$$

while the diameters $D_1'$, $D_2'$, $D_3'$, . . . &c., of the first, second, third, . . . &c., dark rings will have the value—

$$D_1' = 2\sqrt{R\frac{\lambda}{2}},\ D_2' = 2\sqrt{R\frac{3\lambda}{2}},\ D_3' = 2\sqrt{R\frac{5\lambda}{2}}.$$

**To determine the Wave-length of Sodium Light by means of Newton's Rings.**—

EXPT. 64.—The convex surface of a lens (of which the radius of curvature has been determined by the spherometer or by one of the optical methods described in Chap. V.) is laid on a piece of plate-glass so as to form an air film. A Bunsen flame, which traverses an iron wire ring loaded with common salt, is placed in the focal plane of a condensing lens, and the light transmitted through the lens is thrown downwards on to the air film by reflection from an inclined sheet of plate-glass (Fig. 216). The rings are viewed by the aid of a travelling microscope. Measure the diameters of as many bright rings as possible. If $D_n$ is the diameter of the $n$th bright ring, and $\lambda$ is the wave-length of the sodium light, then—

FIG. 216.—Method of observing Newton's Rings.

$$\lambda = \frac{D_n^2}{2R(2n + 1)}.$$

EXPT. 65.—Prove that the diameters of successive bright and dark rings are in the proportions given on p. 409.

EXPT. 66.—Substitute a luminous gas flame for the sodium flame used in Expt. 64. Measure the diameters of the bright rings, when viewed (1) through red, and (2) through blue glass. Obtain the ratio of the mean wave-lengths transmitted by blue and red glass respectively.

**Rings formed by Reflected Light, with White Centre.**—Let us now suppose that the film (Fig. 213) separates media of different refractive indices, the index of the film being less

than that of the medium above it, but greater than that of the medium beneath it. Then each of the rays BC and DE will be formed by reflection at a denser medium, and a phase change amounting to $\pi$ will occur in each case. If the thickness of the film is very small in comparison with the wave-length of light, the film will appear bright by reflected, and black by transmitted, light.

The refractive index of oil of sassafras is intermediate between the refractive indices of crown and flint glasses. Dr. Young found that when oil of sassafras is enclosed between lenses of crown and flint glass, the point of contact of the lenses is surrounded by a *white* spot when viewed by reflected light, and by a *black* spot when viewed by transmitted light.

**To obtain Broad and Bright Rings.**—When the convex surface of a lens is placed in contact with a plane glass surface, the thickness of the air film increases rapidly as we proceed from the point of contact toward the periphery of the lens, unless the radius of curvature of the convex surface is very large. The result of this is that the rings appear very narrow. As the air film is viewed more obliquely, the diameters of the rings increase, since at a point where the thickness is equal to $\delta$, the distance retardation will have the value $2\delta \cos r$ (p. 407), where $r$ is the angle of refraction of the ray into the film. Thus, as $r$ increases, the value of the retardation at a particular point diminishes. However, the rings become very faint when viewed obliquely, owing to the small amount of light which reaches the air film, most of it being reflected from the upper face of the lens. This disadvantage can be overcome by forming an air film between the face of a prism and a lens (Fig. 217). Light entering through one face of the prism falls on the air film bounded by the second face, and is thence reflected to the eye through the third face.

FIG. 217.—Method of obtaining Broad and Bright Rings.

By moving the eye, a position can be found such that the light arriving at it has been reflected at the critical angle from the lower face of the prism. At this point the coloured rings surrounding the central black spot vanish. The reason of this

is, that light no longer penetrates to any great depth into the air film, and so there is only a single ray directly reflected from the upper surface of the latter. The black spot itself remains, thus showing that **a disturbance penetrates the film to a small depth,** and forms an internally reflected ray, capable of producing interference, when the thickness of the film is very small. This result was anticipated from the wave theory (p. 308).

**Perfect Blackness of the Central Spot of Newton's Rings.**—It has been proved experimentally that a film of infinitely small thickness appears perfectly black by reflected light. To account for this in terms of interference, the amplitude of the directly reflected waves must be exactly equal to that of the internally reflected waves. Now the reflected ray BC (Fig. 213) will possess a greater amplitude than the internally reflected ray EF, and thus the interference of these rays could not produce complete blackness. Poisson pointed out that EF will not be the only internally reflected ray. As a matter of fact, an infinite series of rays due to multiple internal reflections will be formed. Fig 218 will serve to explain the formation of the series of internally reflected rays, $\alpha, \beta, \gamma, \delta, \ldots$ all of which will be parallel to BC, the directly reflected ray, if the film is of uniform thickness, and will practically coincide with BC if the film is thin. If the thickness of the film is very small in comparison with the wave-length of light, the retardation due to the differences in the paths of these rays will be negligible. The rays $\alpha, \beta, \gamma, \delta, \ldots$ have respectively experienced 1, 3, 5, 7, . . . internal reflections. Any ray, $\gamma$, has experienced two internal reflections more than the preceding ray, $\beta$, and thus the phases of all will be equal, or will differ by multiples of $2\pi$, which is equivalent to no difference in phase. Consequently, when the film is infinitely thin, the phases of the rays $\alpha, \beta, \gamma, \delta, \ldots$ will all be equal, and will differ by $\pi$ from the directly reflected ray BC. Thus, if the sum of the amplitudes of the rays $\alpha, \beta, \gamma, \delta, \ldots$ is equal to the amplitude of the ray BC, complete interference

FIG. 218.—Rays internally reflected from a Film.

will occur, and the film will appear perfectly black by reflected light.

Sir Geo. Stokes proved, in the following manner, that these conditions are fulfilled. Let $a$ be the amplitude of the incident ray (Fig. 219). The amplitude of the reflected ray will be equal to some fraction of $a$, say $ab$, where $b$ is less than unity. Similarly, the refracted ray will possess an amplitude equal to $ac$, where $c$ is less than unity. If we now reverse the reflected and refracted rays, they must together produce a ray of amplitude equal to $a$.

FIG. 219.—Illustrates Sir G. Stokes's investigation.

The ray $ab$ reversed will give rise to a reflected ray of $b$ times its amplitude (*i.e.* equal to $ab^2$), travelling along the same straight line as $a$; together with a refracted ray of $c$ times its amplitude (*i.e.* equal to $abc$) equally inclined to the surface with $ac$.

Let the ray $ac$ reversed give rise to an internally reflected ray of amplitude $ace$, together with a ray of amplitude $acf$, refracted out into the air along the same straight line as $a$. Thus (Fig. 219), the reversal of the rays $ab$ and $ac$ gives rise to the two rays $acf$ and $ab^2$ travelling along the same straight line as $a$; together with the two rays $ace$ and $abc$, travelling along one straight line within the medium.

Since the reversal of $ab$ and $ac$ must merely reproduce $a$, we have—

$$acf + ab^2 = a; \ \therefore \ cf + b^2 = 1. \quad \dots \quad (1)$$

$$ace + abc = 0; \ \therefore \ e + b = 0. \quad \dots \quad (2)$$

From (2) $e = -b$. Thus, the coefficients of internal and external reflection are numerically equal, but of opposite signs. The explanation of this is, that a phase change of $\pi$ occurs at external reflection, while no phase change occurs at internal reflection.

Returning to Fig. 218, if $a$ is the amplitude of the ray AB, that of the directly reflected ray BC will be $ab$. The ray $\alpha$ was derived from AB by one refraction into the medium, one internal reflection, and one refraction into the air. Thus, its amplitude is equal to $acef$, or, by (1) and (2), to $\{-ab\ (1-b^2)\}$. The ray $\beta$ differs from $\alpha$ only in having suffered two extra internal reflections; its amplitude will be $ace^3f$, or $\{-ab^3\ (1-b^2)\}$. Similarly, the amplitude of the ray $\gamma$ is equal to $\{-ab^5\ (1-b^2)\}$, and so on with the other rays due to multiple internal reflections.

The sum of the amplitudes of the rays due to internal reflections is equal to the infinite series—

$$-ab(1-b^2)\{1 + b^2 + b^4 + \dots\}. \quad \dots \quad (3)$$

The series within the brackets constitutes a geometrical progression, of which the first term is 1, and the common ratio is $b^2$. The sum of $n$ terms of this series is equal[1] to $(b^{2n}-1)/(b^2-1)$. As $n$ increases, the value of $b^{2n}$ continuously diminishes, since $b<1$. Thus, when $n=\infty$, the sum of the series within the brackets in (3) is equal to $1/(1-b^2)$, and the value of (3) reduces to $-ab$. This value is equal to the amplitude of the directly reflected ray, BC, with sign reversed. Consequently, the series of rays due to multiple internal reflection in the film together just annul the directly reflected ray BC, and the perfect blackness of the central spot in Newton's rings is accounted for.

It may be left as an exercise to the student to prove that the sum of the amplitudes of all the transmitted rays (Fig. 218) is equal to $a$, the amplitude of the incident ray.

**Test of Plane Surfaces.**—Let an air film be formed between two surfaces in close proximity. If both surfaces are accurately plane but slightly inclined to each other, interference bands, parallel to the line in which the surfaces would intersect, are observed when the film is illuminated by light from a Bunsen flame coloured with a sodium salt. If one surface only is plane, the bands formed will be curved, and each band passes through all points of the air film which are of uniform thickness. Thus, the bands form contour lines for the curved surface. By this means, having given one plane surface, the planeness of any other surface can be tested. Lord Rayleigh has tested the planeness of a glass surface by supporting it, at a distance of about half a millimetre, below the surface of water. The water surface will be truly plane if care is taken to remove floating specks of dust, &c.; the nature of the glass surface can be inferred from the shape and disposition of the interference bands formed under monochromatic illumination.

**Interference produced by Thick Films.**—When the thickness of a film exceeds a few wave-lengths of light, no interference will be observed if white light is used as an illuminant. The overlapping of the various sets of fringes produces an illumination sensibly uniform. Using monochromatic light as an illuminant, bands can be seen on comparatively thick films. Thus, thin sheets of mica, glass bubbles, &c., will be seen to be covered with alternate bright and dark bands, when viewed by

[1] Hall and Knight's *Higher Algebra*, p. 39.

the light from a Bunsen flame coloured with sodium. Using a mercury vacuum lamp, bands can be observed on a sheet of glass several centimetres in thickness. The shape and disposition of these bands indicate whether the bounding surfaces of the plate are plane and parallel.

**Spectroscopic Examination of Interference Fringes.**—Let an image of a set of interference fringes be focussed on the plane of a spectroscope slit, the bands being parallel to the slit. The resulting spectrum will exhibit dark bands corresponding to the wave-lengths of light which interfere at the position of the slit. If the interference is due to a thin film, then a single dark band will occupy the whole of the spectrum when the film is infinitesimally thin, and its image is formed by reflected white light. If the image of the film is shifted so that thicker portions are focussed on the slit, violet light will first appear, then blue, yellow, and red. We shall soon, however, arrive at a thickness where two different wave-lengths are simultaneously destroyed by interference; in this case there will be two dark bands in the spectrum. For a greater thickness the dark bands will be more numerous.

Let $\delta$ be the thickness of the part of the film which is focussed on the slit, and let $\mu_1$ be the refractive index of the film for red light. Then, if the light is reflected normally from the film, and $\lambda_1$ is the wave-length corresponding to a dark band toward the red end of the spectrum, $2\mu\delta = n\lambda$, where $n$ is an integer (p. 407). The internally reflected red ray is retarded by $n$ wave-lengths behind the directly reflected ray. As we pass toward the blue end of the spectrum, we may arrive at a position such that the internally reflected waves, being shorter than those of the red light, are retarded by $(n+1)$ wave-lengths behind the directly reflected waves. Here, there will be another dark band in the spectrum. If $\lambda_2$ is the wave-length of the light destroyed, and $\mu_2$ is the corresponding refractive index of the film, we shall have $2\mu_2\delta = (n+1)\lambda_2$. Thus, the wave-lengths $\lambda_1$, $\lambda_2$, $\lambda_3$, . . . &c., corresponding to the various dark bands, will be given by the equation—

$$2\delta = n\lambda_1/\mu_1 = (n+1)\lambda_2/\mu_2 = (n+2)\lambda_3/\mu_3 = \ldots$$

As the thickness of the film progressively increases, the dark bands are seen to move toward the red end of the spectrum, new ones being formed toward the violet end. At the same time the bands become narrower and more numerous. Ultimately the bands become too

narrow to be observed. The black bands in the spectrum will be visible long after any trace of colour can be discerned by the eye in the light reflected from the film.

EXPT. 67.—Illuminate the slit of a spectrometer with white light, and cover the slit with a thin film of mica. Notice the dark bands in the spectrum. Prove theoretically that the wave-lengths $\lambda_1, \lambda_2, \lambda_3, \ldots$ corresponding to the dark bands are given by—

$$2\delta = \left(n + \frac{1}{2}\right)\lambda_1/\mu_1 = \left(n + \frac{3}{2}\right)\lambda_2/\mu_2 = \left(n + \frac{5}{2}\right)\lambda_3/\mu_3 = \ldots$$

Note that in this case the bands are formed by transmission (p. 407).

**Situation of Interference Fringes.**—On observing a very thin film, such as a soap bubble, the interference bands appear to coincide with the film; in other words, a motion of one's eye produces no relative displacement between the bands and the film, or there is no parallax (p. 23). When, however, the film is thick, the bands do not in general appear to be situated on it; their formation in this case must now be considered.

FIG. 220.—To determine the Situation of Interference Fringes.

Let KL, MN (Fig. 220) represent sections of the plane bounding surfaces of a film, perpendicular to the paper. Let LK and NM meet toward the left at a small angle, $\theta$. The ray AB, incident at an angle $i_1$ on MN, gives rise to the reflected ray BP; together with the ray BC, refracted, at an angle $r_1$, into the film. BC is incident at an angle $i_2$ on the surface LK, and forms the reflected ray CD. The ray CD is incident at an angle $i_3$ on the surface MN, and gives rise to the ray DP, refracted, at an angle $r_3$, into the atmosphere. Interference or reinforcement occurs at the point P, where the rays BP and DP intersect. If, after leaving P, the rays DP and BP fall on a lens, they will once more cross, and reinforce or interfere with each other, at the focus conjugate to P. Thus, P will be the position at which the interference band appears to be situated.

With P as centre, and PD as radius, describe the arc DG. Produce BC to F, making CF equal to CD. With F as centre, and radius FD, describe the arc DK. The lines DK and DG will be approximately straight, and perpendicular to BC and BP respectively.

The portions DP and GP of the paths of the interfering rays are equal. Also BG = BD sin $i_1$, and BK = BD sin $r_1$. Thus, since the distance BK in the film is equivalent to $\mu$. BK = $\mu$. BD sin $r_1$ in air, and sin $i_1 = \mu$ sin $r_1$, the portions BG and BK of the paths of the interfering rays are equivalent. Thus, the distance of retardation of the ray DP is equal to $\mu(\mathrm{KC} + \mathrm{CD}) = \mu\mathrm{KF}$. Join FD. Then FD is perpendicular to KL, and equal to $2\delta$, where $\delta$ = DH; also $\angle \mathrm{KFD} = \angle \mathrm{BCD}/2 = i_2$. Then $\mu\mathrm{KF} = \mu \times \mathrm{DF} = 2\mu\delta$ (compare p. 406).

Let DP = $\alpha$, while $\angle$ BPD = $\phi$. Then—

$$\frac{\mathrm{DP}}{\mathrm{BD}} = \frac{\alpha}{\mathrm{BD}} = \frac{\sin \mathrm{PBD}}{\sin \mathrm{BPD}} = \frac{\sin\left(\frac{\pi}{2} - i_1\right)}{\sin \phi} = \frac{\cos i_1}{\sin \phi}.$$

$$\frac{\mathrm{FD}}{\mathrm{BD}} = \frac{2\delta}{\mathrm{BD}} = \frac{\sin \mathrm{FBD}}{\sin \mathrm{BFD}} = \frac{\sin\left(\frac{\pi}{2} - r_1\right)}{\sin i_2} = \frac{\cos r_1}{\sin i_2}.$$

$$\therefore \quad \frac{\alpha}{2\delta} = \frac{\sin i_2}{\sin \phi} \cdot \frac{\cos i_1}{\cos r_1}.$$

When $i_1$, the angle of incidence, and $\theta$, the angular inclination of the surfaces of the film, are small, the angles $r_1$, $i_2$, $i_3$, $r_3$, and $\phi$ will all be small. Thus, we may substitute unity for the value of any cosine, and the circular measure for the sine, of any angle occurring in our calculation. Thus—

$$\alpha = 2\delta\frac{i_2}{\phi}.$$

From the triangle DPB—

$$\phi + \left(\frac{\pi}{2} + r_3\right) + \left(\frac{\pi}{2} - i_1\right) = \pi\,; \ \therefore \ \phi = i_1 - r_3 = i_1 - \mu i_3,$$

since the relation $\sin r_3/\sin i_3 = \mu$, reduces to $r_3/i_3 = \mu$.

From the triangle BCD—

$$\left(\frac{\pi}{2} - i_3\right) + 2i_2 + \left(\frac{\pi}{2} - r_1\right) = \pi\,; \ \therefore \ i_3 = 2i_2 - r_1.$$

From the triangle of which BC is the base, and the lines CK and BM produced to meet at an angle $\theta$, are the sides, we have—

$$\theta + \left(\frac{\pi}{2} + i_2\right) + \left(\frac{\pi}{2} - r_1\right) = \pi\,; \ \therefore \ i_2 = r_1 - \theta.$$

Hence—

$$\phi = i_1 - \mu i_3 = i_1 - \mu(2i_2 - r_1) = i_1 - \mu(2r_1 - 2\theta - r_1)$$
$$= i_1 + 2\mu\theta - \mu r_1 = 2\mu\theta,$$

since—

$$i_1/r_1 = \mu.$$

Thus, finally—

$$a = \delta \frac{i_2}{\mu\theta}.$$

SPECIAL CASES.—(1) When $\delta$ is very small, $a$ is approximately equal to 0, unless $\theta$ is exactly equal to 0. This result explains why the interference bands apparently coincide with the film when the latter is very thin.

(2) When $\theta = 0$, $a = \pm \infty$, according as $i_2$ is positive or negative. Thus, when the film has an appreciable thickness and its bounding surfaces are parallel, the bands will be formed at an infinite distance. To view them, the eye must be at rest, or, if a telescope is used, it must be focussed for infinity.

(3) If $\theta$ is small, but not equal to zero, a small difference in $i_2$ or $\delta$ will make a great difference in the value of $a$. Since different portions of the film will be seen by means of rays incident on LK at different angles, at points where $\delta$ has different values, the bands will never be all in focus at once.

When $\theta$ is negative, while $i_2$ is positive, the point P is virtual, and is situated behind the film.

**Michelson's Interferometer.**—An ingenious arrangement for experimenting on the interference of light under great retardation has been invented by Prof. Michelson. Its characteristic features are represented diagrammatically in Fig. 221. A light-ray, LA, is incident at an angle of 45° on the surface of a glass plate thinly coated with silver; the thickness of the silver is such that the intensities of the reflected and transmitted rays, AB and AC, are equal. The rays AB and AC, which are at right angles to each other, are reflected, at normal incidence, from the polished silver surfaces of the mirrors M and N respectively; they then retrace their paths, the ray BA being transmitted, while CA is reflected, at A, after which both follow the path AD. Since the resultant rays are derived from the single ray LA, they will be in a condition to interfere with, or reinforce, each other, according as their

phases differ by an odd or even multiple of $\pi$. The ray reflected from M has previously been reflected at A in air from the thin silver film, and is subsequently transmitted through the latter and the glass plate H supporting it. The ray reflected at C has previously been transmitted through the thin silver film and its supporting glass plate H, and is subsequently reflected in glass from that film. The phase changes on reflection at M and N are similar, and it has been found that, when the silver coating of H is thin, the phase changes on reflection from opposite sides of it, in air and in glass respectively, are similar,[1] each being equal to $\pi$.

FIG. 221.—Optical System of Michelson's Interferometer.

Thus, when the paths AB and AC are equal, or differ by any whole number of wave-lengths, the resulting rays will be in the same phase, and will reinforce each other; when the paths differ by an odd number of half wave-lengths, interference will occur. It will be seen that the ray reflected from N traverses the glass plate H three times in all, while that reflected from M only traverses H once. In order to make the paths of the rays exactly similar, a compensating glass plate, K, of equal thickness with H, and parallel to the latter, is interposed in the path of the ray AB.

On looking along the direction DA, we see the mirror M together with the image of N in the thin silver coating of H. Thus, the interference fringes practically consist of Newton's rings formed at an air film bounded by the silvered surface of M, and the image in H of the silvered surface of N.

In order to obtain an extended source of illumination (p. 408), the

[1] "Phase Change of Light when Reflected at a Silver Surface," E. Edser and H. Stansfield, *Nature*, Sept. 29, 1897.

source of light, S, is placed at the focus of a lens, L. Fig. 221 shows the plane waves derived from a particular point in the source.

To adjust the instrument, the paths AB and AC are first made approximately equal by measurement. A sheet of tinfoil pierced with a pin-hole is then placed over the surface of L, and the two images of the pin-hole, due to the rays ACAD and ABAD respectively, are superposed by adjusting the inclination of N. The image reflected from the unsilvered surface of H can easily be distinguished from that reflected from the thin silver coating. On placing a sodium flame at S, and removing the tinfoil, interference bands are seen. If the paths AB and AC are not quite equal, the thickness, $\delta$, of the equivalent air film will be appreciable, and the bands will be displaced across the mirrors when the eye is moved (p. 418). The mirror M is then slowly moved backward or forward until the parallax between it and the bands vanishes. On substituting a source of white light at S, and slowly moving M backwards or forwards through a small distance, brilliant coloured bands with a white centre make their appearance. The width of these bands can be altered by adjusting the inclination of the mirror N.

**Resolution of Spectral Lines.**—Let us suppose that a source of pure monochromatic light is placed at S. Interference bands, formed in the manner already described, will be seen on looking along the direction DA. If the mirror M is moved backwards through a distance equal to a quarter of a wave-length, the thickness at each point of the equivalent air film will be increased by a quarter wave-length, and at each point where there was previously a bright band, there will now be a dark one. A further motion of M through a quarter wave-length will leave bright bands where there were bright bands at first. In general a displacement of M through any integral number of half wave-lengths will leave a system of bands identical with those originally observed.

Now let us suppose that the source S simultaneously emits waves of lengths $\lambda_1$ and $\lambda_2$, which are nearly, but not quite, equal. When the paths AB and AC are equal, the two sets of bands will occupy practically identical positions. Let the mirror M now be displaced parallel to itself. If $\lambda_1$ is greater than $\lambda_2$, the distance through which the mirror is moved may be simultaneously equal to $n\frac{\lambda_1}{2}$, and to $(n+\frac{1}{2})\frac{\lambda_2}{2}$. In this case bright bands corresponding to $\lambda_1$ will occupy the same positions as when the paths

AB and AC were equal, but a bright band corresponding to $\lambda_2$ will occupy the position originally filled by a dark band, and *vice versâ*. Thus, a dark band due to $\lambda_1$ will be covered by a bright band due to $\lambda_2$, and the fringes will become indistinct, or vanish. On moving M through twice the above distance, it will be displaced through $2n.\frac{\lambda_1}{2}$, or $2(n+\frac{1}{2})\frac{\lambda_2}{2} = (2n+1)\frac{\lambda_2}{2}$. The bright bands due to $\lambda_1$ and $\lambda_2$ will now coincide, and the fringes will be distinctly visible. As the mirror M is further displaced, the bands will become alternately distinct and indistinct. We thus have a means of distinguishing between light which is purely monochromatic, and that which consists of two or more wave-lengths so nearly equal that they cannot be resolved by a spectroscope.

By means of his interferometer Michelson has succeeded in resolving a number of spectral lines which were previously thought to be homogeneous. Light from a vacuum tube was analysed by a prism in the ordinary manner, and one of the spectral lines was thrown on a slit in a diaphragm placed at S (Fig. 221). The equivalent air film, in some experiments, had a great thickness (in some it amounted to 20 cms. or more); thus, the difficulty had to be met that all of the bands are not in focus at once. This was done by using perfectly plane mirrors, adjusted so that the boundaries of the air film were strictly parallel ($\theta = 0$ in the formula for $\alpha$ on p. 418) when the bands were all formed at infinity; in this case they take the form of circles, and must be viewed by means of a telescope focussed for infinity. The mirror M was displaced by means of an accurate screw, and the alterations which occurred in the visibility of the bands were observed.

Michelson found that each of the D lines is itself double, as is also the red hydrogen C line ($\lambda = 6563$), and many others not previously resolved. The green mercury line ($\lambda = 5461$) was found to consist of six lines very close together. Interference was obtained with the green light from a mercury vacuum lamp when the difference in path of the interfering rays amounted to 40 cms.! The red cadmium line ($\lambda = 6439$) was found to be the most homogeneous readily obtainable.

**The Metre in Terms of Wave-lengths of Light.**—Modern scientific measurements are all based on the metre as a standard of length. It is important that we should have some means of accurately reproducing the standard metre preserved at Sèvres, in case some accident should happen to it. Maxwell suggested

that the wave-length of light, emitted by a suitable chemical element under prescribed conditions, would form the best standard of length. To carry out this idea, Michelson has determined how many wave-lengths, corresponding to the red cadmium line, are equivalent to the standard metre. The number of interference fringes which passed across a given point, as the mirror M was moved through a measured distance, was observed. He found the standard metre to be equivalent to 1,553,163·5 wave-lengths of the red cadmium line. According to this measurement, which is one of the most accurate ever performed, the wave-length of the red cadmium line is equal to 6438·5722 tenth-metres. The error in this result probably does not exceed one in a million.

**Jamin's Interferometer.**—In this instrument, interference fringes are produced by a method originally due to Brewster. A ray, AB (Fig. 222), incident on a thick glass plate at an angle of 45°, is split up into a directly reflected ray, BE, together with a refracted ray, BC; the latter giving rise, by internal reflection at C, and transmission into the air at D, to the ray DF, parallel to BE. BE and DF are then incident on a second glass plate similar and parallel to the first one. A component of the ray BE is refracted into this plate at E, internally reflected at D, and refracted into the air at F; the resulting ray, FH, coincides with the component of DF which is directly reflected at F. If the two glass plates are exactly similar and parallel, the two paths traversed by the light will be equivalent; but by slightly tilting one of the plates a difference of path may be introduced, and interference bands will be formed. An extended source of light must be used (p. 408).

FIG. 222.—Optical System of Jamin's Interferometer.

If similar tubes containing different gases are placed so as to be traversed by the rays BE and DF respectively, a measurement of the shift produced in the bands can be used to compare the refractive indices of the gases. If one tube is exhausted, the refractive index of the gas contained in the other can be determined. Profs. Reinold and

Rücker used Jamin's interferometer to determine the thickness of black soap films. From p. 403 we know that the thickness of these is small in comparison with the wave-length of light; consequently, a number must be interposed in the path of one of the interfering rays in order to obtain an appreciable shift of the bands. The end of a glass tube was dipped into a soap solution, and the tube was inverted till the soap film formed slid a short distance into the tube. A second film was then formed, and the above procedure was repeated till the tube enclosed about sixty films. The tube was then placed in the path of one of the interfering rays of the interferometer, and the shift of the bands, when the films appeared black by reflected light, was measured. From this result, combined with a knowledge of the number of films, and the refractive index of the soap solution, the average thickness of the films was calculated (p. 392). This was found to be equal to about $10\mu\mu$ ($10 \times 10^{-6}$ mm.). It was found that an abrupt change in the thickness of the film occurs at the point where it becomes black. This apparently indicates that the thickness of a black soap film is comparable with the diameter of a molecule. It is probable that the diameter of a molecule lies between 0·5 $\mu\mu$ and 0·005 $\mu\mu$.

**Lippmann's Colour Photography.**—When light is reflected normally from a perfectly reflecting surface, the incident and reflected waves combine to form stationary waves (p. 263). If a phase change of $\pi$ occurs on reflection, there will be a node at the surface, and other nodal planes will be found at distances equal to $\lambda/2, \lambda, 3\lambda/2, 2\lambda, \ldots$ from the surface. In each of these planes the ether will be stationary; midway between any two of them (at an antinode) the ether will alternately suffer great displacement, and acquire great velocity (p. 264). If the reflecting surface is covered by a layer of a transparent substance, stationary waves will be formed in the ether penetrating the latter, and the displacement of the ether at the antinodes will there produce an oscillatory motion of the material particles (p. 278), while at the nodes the particles will be at rest. Thus, if chemical change is produced by the light, it is natural to anticipate that, under the above conditions, this will occur only at the antinodes. The layer covering the reflecting surface would then comprise a number of equidistant planes in which chemical change is produced, separated by spaces which remain unacted upon.

To test this point, Prof. Lippmann placed the film of a

photographic plate in contact with clean mercury, and exposed it to light on the glass side. The stationary waves formed in the film acted on the silver salts only in the planes passing through the antinodes. After development the undecomposed silver salts were dissolved out in the usual manner. When dry, the film comprised a number of transparent layers of gelatine, separated from each other by thin silver films. The thickness of each transparent layer was equal to half the wave-length of the light used during the exposure. When illuminated by white light, incident normally, partial reflection occurred at each of the thin silver films. The waves, reflected from the two silver films bounding any particular transparent layer, suffered

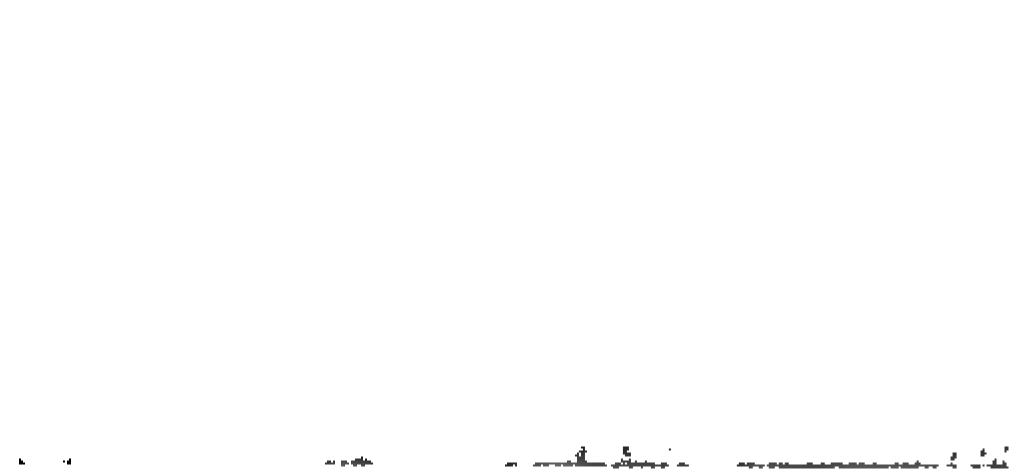

FIG. 223.—Transverse Section of Lippmann Film (magnified about 1,500 Diameters).

similar phase changes on reflection, and thus their final phase difference was due merely to the difference in their paths. Waves equal in length with those used during the exposure were totally reflected, since, for them, the retardation in a transparent layer amounted to a complete wave-length. Other waves penetrated the photographic film and were absorbed. Thus, the film, when viewed normally by the aid of white light, appeared of the same colour as the light to which it had previously been exposed. When the film was viewed obliquely, the wave-length of the reflected light became less as the angle of reflection was increased (p. 407).

Prof. Lippmann also focussed a pure prismatic spectrum on a film under the conditions previously described, and obtained

a photograph of the spectrum in its natural colours. Coloured objects have also been photographed in a similar manner.

The first investigator to obtain microscopic evidence of the laminated structure of Lippmann films was Dr. Neuhauss. Fig. 223 is a reproduction of a photomicrograph obtained in the following manner. Mr. E. Senior made a photograph of the spectrum, and then stripped the film from its glass support. Mr. W. B. Randles made very thin transverse sections of this film, and Mr. T. A. O'Donohoe, by magnifying one of these sections 1,000 diameters, produced the photomicrograph of which Fig. 223 is an enlarged reproduction. The upper surface was the one which was in contact with the mercury. The dark bands represent the antinodal planes, where the silver salts have been decomposed by the light; these are separated by clear spaces, representing the nodal planes. The film was originally exposed to red light, so that the distance between two adjacent antinodal planes was equal to about 0·0035 mm. Fig. 223 is of great interest, since it is impossible, by means of any form of microscope, to see objects *much* smaller than the wave-length of light; the success obtained in this case is partly due to the circumstance that the treatment of the stripped film previous to cutting caused it to swell considerably.

## Questions on Chapter XVI

1. Describe the method of determining the wave-length of a given source of monochromatic light by the bi-prism, and explain how the necessary adjustments are made and tested.

2. Discuss the observed phenomena of interference in relation to the doctrine of the conservation of energy.

3. A plane soap film, illuminated by white light, gradually becomes thinner as the liquid drains away. It is viewed through a spectroscope, the slit of which is horizontal, and which always is directed to the same part of the film. Describe and explain the phenomena which are observed.

4. Explain the colours seen when a thin film of oil is spread over the surface of water.

5. Describe Newton's rings as seen by reflection. What relations hold between the thickness of the air space, the diameters of the rings, and the angle of incidence of the light?

6. Discuss the method of determining the wave-length of light from observations on Newton's rings.

7. Newton's rings are formed between a plane surface of glass and a lens. The diameter of the third black ring is 1 cm. when soda light (wave-length = $589 \times 10^{-7}$ cm.) is used at such an angle that

the light passes through the air film at an angle of 30° to the normal. Find the radius of the glass lens.

8. Describe the method of producing interference of light by the reflection of a beam of light from the front and back surfaces of thick plates of glass. How may such an arrangement be employed to determine the change in the refractive index of a liquid with change of temperature?

9. Describe some method of producing interference fringes in which the difference of path between the interfering pencils is considerable, and point out the conditions for distinctness in the fringes. Show how your method could be used to measure small changes in the refractive index of a body.

10. Newton's rings are formed in sodium light between a flat lens and a plane: as the distance between the lens and the plane is increased, the rings disappear and reappear periodically. Explain this, and show how the phenomenon may be used to analyse the nature of a bright line in the spectrum.

11. Write a short essay on the accurate determination of the wave-length of light.

## Practical

1. Measure the apparent diameter of Newton's rings (in sodium light), using light incident at different angles, and plot the relation between incidence and diameter.

2. Determine, by means of Newton's rings, the wave-length of the light transmitted by the given coloured glass.

3. Project Newton's rings on to a screen, and by measuring them determine the ratio of the wave-lengths of the three given coloured lights.

4. Compare the wave-length of the light transmitted by red glass with that of soda light, by means of Fresnel's bands.

5. Determine the curvature of the surface of a lens by means of Newton's rings. The wave-length of sodium light is $5892 \times 10^{-8}$ cm., and $R = \frac{1}{n\lambda}(r_2^2 - r_1^2)$, where $r_1$ and $r_2$ are the radii of the $x^{th}$ and the $(x + n)^{th}$ dark rings as seen by transmitted light.

6. Arrange Fresnel's bi-prism on the optical bench so as to exhibit the interference bands as well as possible. Describe the adjustments you make, and indicate why you make them.

# CHAPTER XVII

## DIFFRACTION

**Introductory.**—A satisfactory explanation of the rectilinear propagation of light has already been obtained in terms of the Wave Theory (Chap. XIII). The reasoning used shows that wave propagation is approximately rectilinear, when the length of the waves is small. It has also been proved, by independent methods, that, as a matter of fact, the wave-length of light varies between $0·4 \times 10^{-3}$ mm. and $0·8 \times 10^{-3}$ mm., according to its colour. Theory indicates that waves of such lengths should exhibit a slight tendency to bend round corners, and the effects of this should be observable under appropriate conditions. Such effects have been observed, and must now claim our attention. They are classified under the head of **Diffraction Phenomena**. As will be seen, these phenomena are exhibited when part of a wave front is intercepted by one or more opaque obstacles.

**Cylindrical Waves, Half-Period Elements.**—Let S (Fig. 224) be the section of a narrow slit perpendicular to the plane of the paper. Light transmitted through this slit will consist of cylindrical waves of which the axes pass through S. Let the circle APB be the section of an imaginary cylinder with axis passing through S, perpendicular to the plane of the paper.

FIG. 224.—Half-Period Elements.

Each wave from the slit S will in turn pass through the surface APB. The illumination at a point O is due to the combined action

of the wavelets formed at different points of the surface APB. As these points are at different distances from O, the various wavelets which simultaneously arrive at O must have started at different times. The point P, where APB is cut by the straight line OS, is nearer to O than any other point on APB; P is termed the **pole of the wave surface** APB. Let $\lambda$ be the wave-length of the light transmitted through S. With O as centre, and $(OP + \lambda/2)$ as radius, describe arcs cutting APB in $M_1$ and $N_1$. $PM_1$ will be the section of a narrow strip on the cylindrical surface APB. Cylindrical wavelets from the edge $M_1$ of this strip must have started half a period earlier than those from P, in order to arrive at O simultaneously with the latter. Wavelets from the strip $PM_1$ will reinforce each other at O. $PM_1$ is termed the **first half-period element** of the surface PB. With O as centre, and $(OP + \lambda)$ as radius, cut APB in $M_2$ and $N_2$. Cylindrical wavelets from the strip of which $M_1M_2$ is the section must, on an average, have started half a period before those from $PM_1$, in order to arrive at O simultaneously with the latter. The phase of the disturbance from $M_1M_2$ will differ by $\pi$ from that due to $PM_1$. The strip $MM_1$ constitutes the **second half-period element.** With O as centre, and radii equal to $(OP + 3\lambda/2)$, $(OP + 2\lambda)$, $(OP + 5\lambda/2)$, ... cut APB in $M_3$, $M_4$, $M_5$, ... $N_3$, $N_4$, $N_5$, thus dividing PA and PB into successive half-period elements. The resultant disturbance at O is due to the combined effects of the wavelets from the various half-period elements.

EXPT. 68.—Cut a circular disc, of 10 cms. radius, from stout drawing paper. Fasten this at its centre, S (Fig. 224), with a single drawing pin, above a sheet of paper strained on a drawing board. Join S to O, a point 30 cms. from S, by a straight line. Then OP = 20 cms. To obtain a wave model for $\lambda = 0{\cdot}2$ cm., mark off points $M_1$, $M_2$, $M_3$, ... on the edge of the circular disc at distances respectively equal to 20·1, 20·2, 20·3, ... cms. from O; the points $N_1$, $N_2$, ... are found by making $PN_1 = PM_1$, $PN_2 = PM_2$, ... &c. Then P is the pole of the wave surface with respect to the point O, and $PM_1$, $M_1M_2$, ... &c., are the half-period elements due to the half PB of the wave surface, while $PN_1$, $N_1N_2$, ... &c., are the half-period elements due to the other half, PA. Draw TOR perpendicular to SO. To determine the half-period elements with respect to any point Q in TR, rotate the paper disc until the pole P lies in the straight line joining SQ.

It is convenient to deal separately with the halves PA and PB of the wave surface. As a result of the above exercise, it becomes evident that the breadth $PM_1$ of the first half-period element is much greater than that of any succeeding element, while $M_1M_2$, $M_2M_3$ . . . are in descending order of magnitude. Since the elements are all of equal lengths, the areas of the 1st, 2nd, and 3rd, &c., elements must be in descending order of magnitude. Since the rate of decrease is continuous, the area of any element is approximately equal to the mean of the areas of the preceding and succeeding elements. Consequently, the numerical value of the displacement at O, due to wavelets from any particular element, is equal to half the sum of the displacements due to the preceding and succeeding elements.

Let $d_1$, $d_2$, $d_3$, . . . be the numerical values of the displacements at O due to wavelets from the 1st, 2nd, 3rd, . . . &c., elements. Then, indicating the phase difference between the displacements from odd and even elements by prefixing a minus sign to $d_2$, $d_4$, . . . (compare p. 291), we find that D, the resultant displacement at O due to the half, PB, of the wave surface, is given by—

$$
\begin{aligned}
D &= d_1 - d_2 + d_3 - d_4 + d_5 - \dots \\
&= d_1/2 + \{(d_1 + d_3)/2 - d_2\} + \{(d_3 + d_5)/2 - d_4\} + \dots \\
&= d_1/2.
\end{aligned}
$$

The displacement at O due to the whole wave surface is equal to 2D, or $d_1$.

**Diffraction at a Straight Edge.**—Let the straight edge C and the illuminated slit S (Fig. 225) be parallel to each other, and perpendicular to the plane of the paper. It is required to determine the illumination on a screen TOR, also perpendicular to the paper.

Join SC, and produce to O. A line on the screen through O, perpendicular to the plane of the paper, defines the limit of the geometrical shadow of the straight edge. At O the illumination is due to the half-period elements comprised in one half of the wave surface, and the displace-

FIG. 225.—Diminution of Brightness within the Geometrical Shadow of a Straight Edge.

**ment is equal to $d_1/2$. As we pass along OT into the geometrical shadow, the 1st, and then the 2nd, 3rd, 4th, . . . of the half-period elements are intercepted.**

When the 1st element only is intercepted, the displacement D is given by—

$$\begin{aligned} D &= -d_2 + d_3 - d_4 + d_5 - \dots \\ &= -d_2/2 - \{(d_2 + d_4)/2 - d_3\} - \{(d_4 + d_6)/2 - d_5\} - \dots \\ &= -d_2/2. \quad \dots \dots \dots \quad (1) \end{aligned}$$

As the 2nd, 3rd, . . . elements are intercepted, the displacement assumes the values $(+d_3/2)$, $(-d_4/2)$, . . . &c.

As $d_1$, $d_2$, $d_3$, $d_4$, . . . are in descending order of magnitude, the displacement rapidly and continuously decreases in magnitude as we pass into the geometrical shadow; the illumination, which is proportional to the square of the displacement, diminishes still more quickly. Thus, there is a small amount of illumination within the edge of the geometrical shadow, but this rapidly and continuously diminishes as we proceed, and at a small distance within the edge becomes inappreciable (Fig. 227).

FIG. 226.—Formation of Diffraction Fringes outside the Geometrical Shadow of a Straight Edge.

As we pass along OR, the pole travels from C toward B (Fig. 226), and one after another of the previously intercepted half-period elements becomes exposed. Thus, the illumination at a point Q is due to one complete half of the wave surface, together with a certain number of elements of the other half. The displacement $D_1$, due to half of the wave, is given by $D_1 = d_1/2$. As the 1st, 2nd, 3rd, . . . half-period elements of the second half of the wave surface are exposed, the displacement assumes the values—

$D_2 = d_1/2 + d_1 = 3d_1/2$.
$D_3 = d_1/2 + d_1 - d_2 = a_1/2$ (nearly).
$D_4 = d_1/2 + d_1 - d_2 + d_3 = d_1 + d_2/2 + \{(d_1 + d_3)/2 - d_2\} = d_1 + d_3/2$.
$D_5 = d_1/2 + d_1 - d_2 + d_3 - d_4 = d_1/2$.
$D_6 = d_1/2 + d_1 - d_2 + d_3 - d_4 + d_5$
$\quad = d + \{(d_1 + d_3)/2 - d_2\} + \{(d_3 + d_5)/2 - d_4\} + a_5/2 = d_1 + d_5/2$.

The illumination which would be produced at any point if the whole wave were operative is proportional to $d_1^2$. The illumination corresponding to $D_2$ is proportional to $9d_1^2/4$, or $2{\cdot}25d_1^2$. It is therefore greater than that which would be produced at the

point by the unobstructed wave. The illumination corresponding to $D_3$ is proportional to $(d_1/2)^2$, and is thus less than that corresponding to $D_2$. $D_4$ corresponds to an illumination slightly less than that due to $D_2$, since $d_3 < d_1$, but greater than that due to $D_3$, while $D_5$ corresponds to an illumination less than that due to $D_4$, and so on. Thus, as we proceed along OR, away from the edge of the geometrical shadow, a number of bright bands separated by comparatively dark intervals are encountered (Fig. 227). Owing to the unequal widths of the half-period elements, the bright bands on the screen occur at unequal intervals, the first band being widest, while the rest decrease in regular succession. The distinctness of these bands becomes less and less as we proceed, and after a time the illumination becomes uniform.

FIG. 227.—Diffraction Bands bordering the Shadow of a Straight Edge.

**Diffraction Bands can only be observed when the Source of Light is of very Small Dimensions.**—An illuminated pinhole or narrow slit is generally employed. An extended source of light is equivalent to a large number of linear sources, and these give rise to different sets of bands, which overlap and produce uniform illumination.

For a given disposition of the source, straight edge, and screen, the width of corresponding half-period elements will be smaller for blue than for red light. As a consequence, the width of the bands will be less for blue than for red light. When white light is used, the inner edges of the first few bands will be blue, and the outer edges red.

**Narrow Obstacle.**—A fine wire or other similar obstacle, when placed parallel to an illuminated slit, intercepts some of the wave elements, and thus produces diffraction effects. On either side of the geometrical shadow may be observed a series of bands similar to those already described, and produced in a similar manner. Those on either side are produced by the light which has passed the corresponding edge of the obstacle; they are of unequal widths, and are unaffected by the breadth of

the obstacle. Inside the geometrical shadow another series of bands is formed. These bands are narrower than those previously mentioned, and are approximately equidistant from each other, the space between two bright bands being inversely proportional to the breadth of the obstacle.

The displacement at the middle of the geometrical shadow is due to the unintercepted elements in the halves of the wave surface. If the obstacle intercepts only the first elements in both halves, the displacement at the middle of the geometrical shadow will be equal to $2 \times -d_2/2 = -d_2$. This is only slightly smaller than that obtained when the obstacle is removed. If the first and second elements in both halves are intercepted, the displacement at the middle of the geometrical shadow will be equal to $d_3$, and so on. Since $d_1, d_2, d_3, \ldots$ are in descending order of magnitude, it follows that the illumination at the middle of the geometrical shadow becomes more feeble as the breadth of the obstacle increases; it becomes inappreciable when the obstacle intercepts more than the first few of the half-period elements.

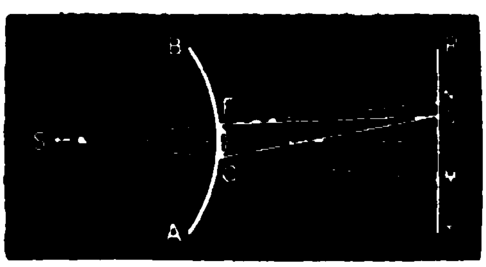


FIG. 228.—Formation of Diffraction Bands within the Shadow of a Narrow Obstacle.

The displacement at any point, Q (Fig. 228), within the geometrical shadow, is due to the combined action of wavelets derived from the wave surface in the immediate neighbourhood of the edges of the obstacle. The wavelets from C and F (Fig. 228) start in the same phase and, on reaching Q, will reinforce, or interfere with, each other, according as (QC – QF) is equal to an even or odd number of half wave-lengths. Thus, if $OQ = x$, and $CF = \delta$, while the distance from O to the point midway between F and C is equal to D, we have, by reasoning precisely similar to that employed on p. 390,

$$x = \frac{D}{\delta} \cdot n\lambda, \text{ [for the } n\text{th bright band to be at Q],}$$

and—

$$x = \frac{D}{\delta} \cdot \left(n + \frac{1}{2}\right)\lambda, \text{ [for the } (n+1)\text{th dark band to be at Q],}$$

where $n$ can be any integer, including 0. The bright band corresponding to $n = 0$ is the central bright band.

The distance between the middle points of two neighbouring bright bands is equal to $D\lambda/\delta$ (compare p. 391). Thus, the smaller we make

the breadth, $\delta$, of the obstacle, the greater becomes the width of a band. With a very narrow obstacle, the internal bands spread out beyond the geometrical shadow.

Fig. 229, for the use of which I am indebted to Mr. W. B. Croft, is a photograph of the shadows of two needles, using light from a narrow slit as an illuminant. The shadow with the wider

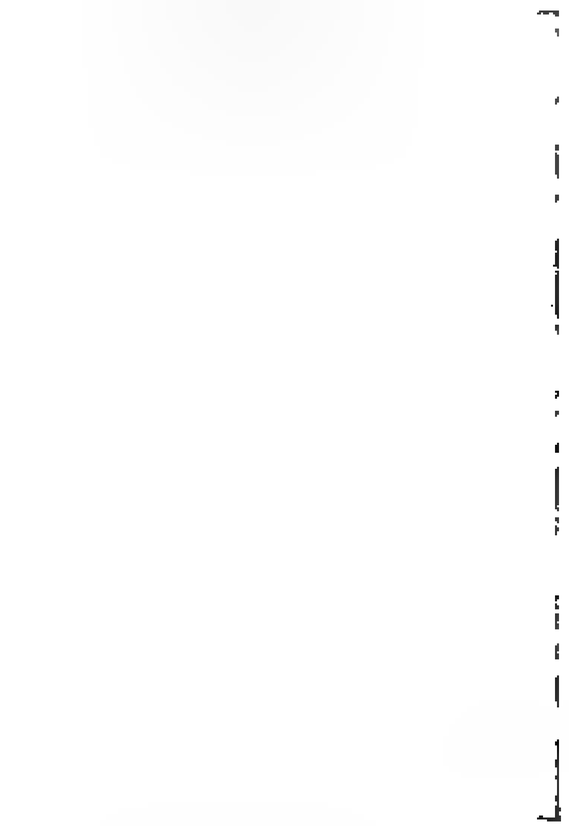

FIG. 229.—Shadows of Pointed Ends of Needles.

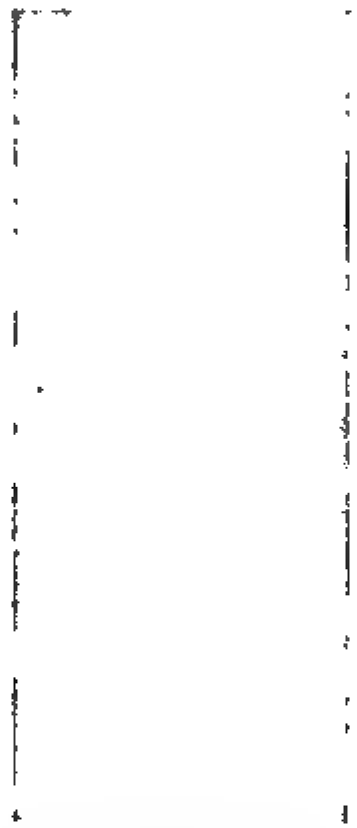

FIG. 230.—Shadow of Eye End of Needle.

bright band at its centre was obtained by using the smallest needle made; the other needle was larger. The outer bands, of unequal widths, are seen, together with the inner equidistant bands. The broadening of the central band towards the point of the needle should be noticed. Fig. 230, also due to Mr. Croft, shows the diffraction bands for the shadow of the eye end of a needle.

EXPT. 69.—Place a convex lens in front of your eye; look through this toward a narrow slit (made as described in Expt. 60, p. 395) placed

in front of a gas flame. On placing a fine wire close in front of the lens, the internal and external diffraction bands are clearly seen.

The wide bands on either side of the bi-prism fringes (Fig. 207, p. 395) are due to a cause similar to that which gives rise to the external bands fringing the shadow of a needle. The bi-prism divides a cylindrical wave from the slit into two portions. The central fringes are formed by interference between wavelets from both portions; the wider external bands on either side are derived only from the half wave surface on that side.

**Rectangular Aperture.**—Let C and F (Fig. 231) represent the edges of a rectangular aperture, while AB represents a cylindrical wave surface of which the axis coincides with the illuminated slit S. If the distance CF comprises a considerable number of half-period elements with respect to the screen TOR, the limits, M and N, of the geometrical shadows of the edges C and F, will be bordered internally by diffraction bands of unequal widths, similar to those described in connection with a straight edge. The illumination of the screen quickly fades out as we proceed to points farther from O than M or N.

Fig. 231.—Diffraction at a Narrow Aperture.

Let us now suppose that CF comprises only a few half-period elements. Let O be a point on the screen equidistant from C and F. If, with respect to O, CF comprises only the first half-period elements of each half of the wave AB, the displacement at O will be equal to $2d_1$, which is a maximum value. If CF comprises the first two half-period elements, the displacement at O will be equal to $2d_1 - 2d_2$, and thus has a minimum value. If CF comprises three half-period elements, the displacement at O is equal to—

$$2d_1 - 2d_2 + 2d_3 = d_1 + d_3 + \{(d_1 + d_3) - 2d_2\} = d_1 + d_3,$$

which is a maximum value, slightly smaller than if the first half-period element were alone comprised by CF. Proceeding in this manner, we see that, using monochromatic light, **the illumination at O has a maximum or minimum value according as the aperture comprises an odd or even number of half-period elements in each half**

of the wave surface. Consequently, if the screen is moved up from a distance toward the aperture, the middle of the illuminated area becomes alternately bright and dark. When white light is employed, the point O will generally be coloured, since CF may comprise an odd number of elements with respect to red, and an even number with respect to blue light.

Let us suppose that CF comprises three half-period elements in each half of the wave surface, with respect to O. There will then be a bright band at O, the displacement there being equal to $d_1+d_3$. As we proceed across the screen toward R, we shall reach a point where the third half-period element of the upper half of the wave is intercepted, while the fourth half-period element of the lower half of the wave is exposed. The displacement at this point is equal to $(d_1-d_2)$ from the upper half of the wave, and to—

$$\begin{aligned} d_1 - d_2 + d_3 - d_4 &= (d_1 + d_3)/2 + \{(d_1 + d_3)/2 - d_2\} - d_4 \\ &= (d_1 + d_3)/2 - d_4 \end{aligned}$$

from the lower half. The resultant displacement is equal to—

$$d_1 + \{(d_1 + d_3)/2 - d_2\} - d_4 = d_1 - d_4,$$

which is a minimum value. At this point there will be a dark band. A little farther on, the third half-period element of the upper half of the wave is intercepted, while the fifth element of the lower half becomes exposed. Here the displacement is equal to $d_1$ from the upper half of the wave, and to—

$$\begin{aligned} &d_1 - d_2 + d_3 - d_4 + d_5 \\ &= d_1/2 + \{(d_1 + d_3)/2 - d_2\} + \{(d_3 + d_5)/2 - d_4\} + d_5/2 = (d_1 + d_5)/2, \end{aligned}$$

from the lower half. The resultant displacement is equal to $(3d_1+d_5)/2$, which is a maximum value. Here there will be a bright band. Thus, the illuminated area MN on the screen is crossed by bands alternately bright and dark.

At a point Q in the geometrical shadow of the edge F, the illumination is due to a limited number of half-period elements belonging only to the lower half of the wave surface. The point Q will be bright or dark according as CF comprises an odd or even number of half-period elements with respect to Q. Thus, when QC – QF is equal to an even number of half wave-lengths, Q will be dark. When QC – QF is equal to an odd number of half wave-lengths, Q will be bright. These relations, it should be noticed, are opposite to those found with respect to a narrow obstacle (p. 432). The bands beyond M and N will be narrower than those within the region MN.

When CF comprises, with respect to O, an even number of elements in each half of the wave, and the central band at O is dark, similar reasoning proves the existence of bands alternately dark and bright in the illuminated region MN, together with finer bands beyond M and N.

Finally, we must consider the case where CF comprises only a small fraction of each first half-period element of the wave surface, with respect to O. There will then be illumination at O, but no bands will be formed within the region MN. There will, however, be well-defined bands external to M and N, formed in the manner already described. If OQ = $x$, while FC = $\delta$, and PO = D, we shall have—

$$x = \frac{D}{\delta}\left(n + \frac{1}{2}\right)\lambda, \quad \text{[for a bright band at Q]},$$

$$x = \frac{D}{\delta} n \,.\, \lambda, \quad \text{[for a dark band at Q]}.$$

The distance between two bright bands is equal to $D\lambda/\delta$, so that a diminution in the width of the aperture increases the width of a band.

Fig. 232, for which I am indebted to Mr. W. B. Croft, shows the appearance of diffraction bands due to very narrow apertures.

Very Narrow Slit. Wider Slit.

FIG. 232.—Diffraction Bands formed by Narrow Slits.

In both cases the apertures were so narrow that only the external bands were formed. These bands are seen to be equidistant, as anticipated by theory. It is also seen that the wider diffraction bands correspond to the narrower aperture.

Fig. 233 is reproduced, on a reduced scale, from a photograph[1] of diffraction bands formed by a narrow tapering slit. The slit used was 15 cms. in length, and tapered from a width of 0·05 mm. at its upper extremity to zero width at its lower extremity. It will be seen that the central bright band, and the fringes which border it, increase in width as the slit becomes narrower.

Fig. 233.—Diffraction Bands formed by Narrow Tapering Slit.

Expt. 70.—Look at the incandescent filament of an electric glow lamp through a narrow slit placed in front of the eye. The diffraction bands are clearly seen. They can be seen by merely looking at the filament through the narrow aperture between two of your fingers.

**Small Circular Aperture.**—In this case an illuminated pin-hole must be used as a source of light, and the waves are spherical. Let S (Fig. 234) be an illuminated pin-hole, and let APB be an imaginary spherical surface with centre at S. Spherical waves, starting from S, will pass through the surface APB in regular succession. Let us divide this surface into half-period elements, with respect to a point O on the screen TOR. Join SO by a straight line; the point P, where SO cuts the surface APB, is the pole of the wave surface. With O as centre, and radii equal to $(OP+\lambda/2)$, $(OP+\lambda)$, $(OP+3\lambda/2)$, . . . describe imaginary spheres cutting the surface APB in circles. The 1st half-period element will be a circular disc surrounding P, and the 2nd, 3rd, . . . elements will be small



Fig. 234.—Half-Period Elements of Spherical Surface.

[1] "Some Diffraction Photographs," by W. S. Franklin, the *Physical Review*, January 1900.

annular strips concentric with P. The central disc and two of these annuli are shown in profile in Fig. 234. The width of an element decreases as we recede from the pole, but the areas of the elements are nearly equal, decreasing but slightly as we recede from the pole (compare p. 290). If $d_1$, $d_2$, $d_3$, . . . represent the magnitudes of the displacements at O due to the 1st, 2nd, 3rd, . . . elements, these magnitudes are nearly equal; the resultant displacement at O due to the whole wave is equal to—

$$d_1 - d_2 + d_3 - d_4 + d_5 \ . \ . \ . \ = d_1/2 \text{ (compare p. 291)},$$

since the displacement due to any element is equal to half the sum of the displacements due to the preceding and succeeding elements.

Let C and F (Fig. 231) represent the edges of a circular aperture. If this aperture is small, the illumination at O will possess a maximum or minimum value, according as the aperture exposes an odd or even number of half-period elements of the wave surface. The proof of this is similar to that given with respect to a narrow rectangular aperture (p. 434). Consequently, if monochromatic light is used, and the screen TOR is moved up toward the aperture from a distance, the point O (Fig. 231) will become alternately dark and bright. If white light is used, the point O will in general be coloured, since the aperture CF may simultaneously comprise an odd number of elements for the red waves, and an even number of elements for the blue waves, or *vice versâ*.

FIG. 235.—Half-Period Elements exposed by Circular Aperture.

At a point on the screen between O and N (Fig. 231), the pole of the wave surface will be excentric with respect to the aperture. Let us suppose that, with respect to O, the aperture comprises four half-period elements, so that the point O is dark. For a certain point at a small distance from O, about half of the fourth element becomes intercepted at one side, and half of the fifth element becomes exposed at the opposite side of the aperture (Fig. 235). The displacement at O was equal to $\{(d_1+d_3)/2-d_4\}$ (compare p. 435), which is a minimum value, since $d_1$, $d_3$, and $d_4$ are nearly equal. At the above-mentioned point on one side of O, the displacement is equal to—

$$d_1 - d_2 + d_3 - d_4/2 + d_5/2 = (d_1 + d_3)/2 - d_4/2 + d_5/2$$
$$= (d_1 + d_3)/2 \text{ (nearly)}.$$

which is a maximum value, and corresponds to a bright band. Since the illumination will be uniform at all points at the same distance from O, the bright band will be circular. At a point farther from O than that already considered, half of the third element will become intercepted at one side, and half of the sixth element exposed at the other side of the aperture. The displacement will then be equal to—

$$d_1 - d_2 + d_3/2 - d_4/2 + d_5/2 - d_6/2 = d_1/2 - d_6/2,$$

which is a minimum value, and corresponds to a dark circular band.

Thus, the point O will be surrounded by a series of circular bands of unequal widths, alternately bright and dark. If the aperture is very large, these bands will only be visible near the limits of the geometrical shadow, as in the case of a straight edge. If the aperture comprises only a few half-period elements with respect to O, the point O will be bright or dark, according to the number of elements comprised in the aperture, and will be surrounded by bands alternately bright and dark (using monochromatic light) or brilliantly coloured (using white light). If the aperture is so small that, with respect to O, only a fraction of the first half-period element is comprised, the point O will be bright, but there will be no bands within the geometrical image of the aperture. With respect to a point Q (Fig. 231) within the geometrical shadow, the aperture exposes a number of short lengths cut off from succeeding half-period elements (Fig. 236). The point Q will be bright or dark, according as the aperture comprises an odd or even number of these fractional elements; in other words, according as QC − QF (Fig. 231) is equal to an odd or even number of half wave-lengths. Thus, when the aperture is exceedingly small, its geometrical image on the screen will be surrounded by a large number of bands alternately dark and bright. The distance between the centres of two neighbouring bright bands, as proved on p. 436, is equal to $D\lambda/\delta$, where $\delta$ is the diameter of the aperture, and D is the perpendicular distance from the centre of the aperture to the screen.

FIG. 236.—Half-Period Elements exposed by Circular Aperture.

Fig. 237, for which I am indebted to Mr. W. B. Croft, is reproduced from photographs of diffraction bands due to four small circular apertures, varying in diameters from 2 mms. to 0·5 mm. In each case the aperture comprised more than one

1 2

3 4

FIG. 237.—Diffraction Bands within the Images of Circular Apertures. (The apertures 1, 2, 3, 4, were in descending order of magnitude.)

half-period element, so that bands were formed within its geometrical image. The centre is dark or bright according to the number of half-period elements comprised.

**Diffraction in Pin-hole Camera.**—It will now be obvious that no advantage can be obtained, with respect to the pin-hole camera, by diminishing the aperture so that it comprises less than one half-period element with respect to the screen. If the aperture is diminished beyond this size, each point of a luminous object will give rise to a broad central spot encircled by rings which extend far beyond the geometrical image of the point, so that the complete image on the screen will become confused (compare Fig. 233).

**Shadow of Circular Disc.**—Since the half-period elements of a spherical wave are nearly equal in area, it follows that there

will be little loss of illumination at O if the first one or two elements are intercepted by an opaque disk. For the illumination at O, due to the whole wave, is proportional to $(d_1/2)^2$; if the first element is intercepted, the illumination is proportional to $(-d_2/2)^2$, while if the first and second elements are intercepted the illumination is proportional to $(d_3/2)^2$. Since $d_1$, $d_2$, $d_3$ . . . are nearly equal, the result stated above follows as a matter of course. This remarkable result was first deduced by Poisson, who considered that it was so far at variance with the recognised properties of light, that it disproved the wave theory by a *reductio ad absurdum*. The occurrence of a bright spot at the centre of the shadow of a small circular disc had nevertheless been observed by Delisle as early as 1715, but had then attracted small attention, and had subsequently been completely forgotten.



FIG. 238.—Shadow of Threepenny-piece, showing Bright Central Spot.

It was thus left for Arago to show that here, as in other cases, the results of experiments are in complete agreement with the predictions of the wave theory of light. Arago found that the shadow of a circular disc 2 mms. in diameter had a bright spot of light at its centre.

Fig. 238, due to Mr. W. B. Croft, is a photograph of the shadow of a threepenny-piece. A pin-hole was illuminated by sunlight, and at a distance of 18 feet a threepenny-piece was suspended by a fine wire. The shadow of the coin was allowed to fall directly on a photographic plate at a further distance of 18 feet. A bright spot is seen at the centre of the shadow.

Under appropriate conditions, the bright spot at the centre of the shadow is seen to be surrounded by fine circular bands, similar in origin to those found within the shadow of a narrow rectangular obstacle (p. 432).

Outside the geometrical shadow are broad diffraction bands of unequal widths, similar to those produced by a straight edge. The diffraction bands due to the fine suspending wire are clearly visible in Fig. 238.

**Zone Plate.**—Let $PM_n$ (Fig. 239) represent an imaginary plane perpendicular to the plane of the paper, and let S be a luminous point in the plane of the paper. Draw SP perpendicular to the imaginary plane, and produce to O. Let us determine the displacement at O due to wavelets produced at points in the imaginary plane, under the action of the spherical waves diverging from S. Let $M_1$ be a point such that the path $SM_1+M_1O$ is equal to $SP+PO+\lambda/2$. A circle surrounding P, of radius equal to $PM_1$, will enclose an area, the wavelets from which reinforce each other on arriving at O. This area constitutes the first half-period zone. Let $M_2$ be a point such that $SM_2+M_2O = SM_1+M_1O+\lambda/2$. With P as centre, and radius $PM_2$, describe a circle on the imaginary plane. Then the annular space, bounded by the concentric circles of radii $PM_1$ and $PM_2$, constitutes the second half-period zone. Wavelets from this zone reinforce each other at O, where they produce a displacement differing in phase by $\pi$ from that due to the wavelets from the first half-period zone. Similarly, let $SM_3+M_3O = SM_2+M_2O+\lambda/2$. Then the annular space, between the circles of radii equal to $PM_3$ and $PM_2$, constitutes the third half-period zone. Proceeding in this manner we can completely divide the plane AB into half-period zones surrounding P.

FIG. 239.—Illustrates the Theory of a Zone Plate.

Let $M_n$ be a point on the external boundary of the $n$th half-period

zone. Then $SM_n + M_nO = SP + PO + n\lambda/2$. Let $PM_n = r$, while $SP = u$, and $PO = v$. Then, if $SM_n = m$,

$$m^2 - u^2 = (m - u)(m + u) = r^2;$$
$$\therefore\ m - u = r^2/(m + u) = r^2/2u,$$

since $m$ differs from $u$ only by a few wave-lengths of light. Thus—

$$SM_n = m = u + r^2/2u.$$

By similar reasoning—

$$OM_n = v + r^2/2v.$$

$$SM_n + M_nO\left(= SP + PO + \frac{n\lambda}{2} = u + v + \frac{n\lambda}{2}\right) = u + v + \frac{r^2}{2}\left(\frac{1}{u} + \frac{1}{v}\right);$$

$$\therefore\ n\lambda = r^2\left(\frac{1}{u} + \frac{1}{v}\right),\ \text{and}\ r^2 = \frac{uv}{u + v}\ .\ n\lambda. \quad . \quad . \quad (1)$$

The values of $r$ for the 1st, 2nd, 3rd, . . . zones can be found by substituting 1, 2, 3, . . . for $n$ in (1). It thus follows that **the external radii of the various zones, for given values of $u$ and $v$, are proportional to the square roots of the natural numbers 1, 2, 3, . . .**

The area enclosed by the $n$th zone is equal to—

$$\pi\ .\ \frac{uv}{u + v}\ .\ \{n\lambda - \overline{n - 1}\ .\ \lambda\} = \pi\ .\ \frac{uv}{u + v}\ .\ \lambda.$$

Since this value is independent of $n$, it follows that **all zones are equal in area.**

Thus, the numerical magnitudes of $d_1$, $d_2$, $d_3$, . . ., the displacements at O due to wavelets from the various zones, diminish only slightly with the order of the zones (compare p. 291). The displacement due to any zone is equal to half the sum of the displacements due to the preceding and succeeding zones, and the resultant displacement at O, due to all of the zones, is equal to $d_1/2$. Let us now intercept the wavelets from the 2nd, 4th, 6th, . . . zones; the resultant displacement at O becomes equal to—

$$d_1 + d_3 + d_5 + d_7 + \ .\ .\ .$$

which is many times greater than that due to the wavelets from all the zones. O will thus be a point of maximum illumination, or, in other words, light from the luminous point S will be brought to a focus at O. The connection between $u$ and $v$, the respective

distances of the object S and the corresponding image O from the zone plate, is given by—

$$\frac{1}{u}+\frac{1}{v}=\frac{n\lambda}{r^2}, \quad \ldots \ldots \ldots \ldots (2)$$

where $r$ is the radius of the $n$th zone. Thus, for a given zone plate, $\frac{1}{u}+\frac{1}{v}=$ a constant value, a result similar to that found for a convex lens. Substituting $u=\infty$, we find that **the focal length of the zone plate** is equal to $r^2/n\lambda$.

We thus find that, with regard to light from a luminous point on the axis, a zone plate acts like a lens. Luminous points at small distances from the axis will also give rise to images at small distances from the axis, so that the similarity between a zone plate and a lens is, so far, complete. But a zone plate has the peculiarity that a number of foci, of decreasing intensity, are situated between it and the brightest focus already mentioned. For, if the values of $u$ and $v$ are such that the first zone comprises three half-period elements, the displacement due to wavelets from that zone has the value $(d_1+d_3)/2$ (p. 434). Wavelets from the 4th, 5th, and 6th elements will be intercepted, while those due to the 7th, 8th, and 9th elements, transmitted by the third zone, will be equal to $(d_7+d_9)/2$, and so on. Thus, the resultant displacement is equal to—

$$\{d_1 + d_3 + d_7 + d_9 + \ldots\}/2,$$

which is greater than that due to the wavelets from all of the zones. In a similar manner it can be proved more generally that a luminous point at a distance $u$ from the plate will give rise to a series of images at distances $v_1, v_2, v_3, \ldots$ determined by the condition that, for each of these values of $v$, a zone comprises an odd number of half-period wave elements.

The positions of the corresponding foci are given by—

$$f_1 = r^2/n\lambda, \quad f_2 = r^2/3n\lambda, \quad f_3 = r^2/5n\lambda, \text{ \&c.}$$

To obtain a zone plate, it is only necessary to draw on paper a large number (about 230) of concentric circles, with radii proportional to the square roots of the natural numbers, and then to blacken alternate zones; a reduced photograph on glass of this drawing, constitutes a zone plate.

Fig. 240 is reproduced from an accurate drawing of a zone plate, executed by Prof. R. W. Wood. If this is reduced by

photography to an external diameter equal to that of a shilling, a zone plate will be formed which is equivalent, with respect

FIG. 240.—Drawing to be used in making a Zone Plate.

to the first and most distinct focus, to a lens of about a metre focal length.

Lord Rayleigh has pointed out, that if, instead of intercepting the wavelets from the 2nd, 4th, . . . &c., zones, we could change the phases of these wavelets by $\pi$, we should obtain at O (Fig. 239) a displacement equal to—

$$d_1 + d_2 + d_3 + d_4 + d_5 + \ . \ . \ .$$

which is about twice as great as that due to the 1st, 3rd, 5th, . . zones, and corresponds to an illumination about four times as great. This has been achieved by Prof. R. W. Wood. A glass plate was coated with a thin layer of gelatine impregnated with bichromate of

potash. Under the action of light bichromated gelatine becomes hard and perfectly insoluble in water. An image of the zone plate (Fig. 240) was formed on the prepared film by the aid of a camera and lens, as in ordinary photography. The portions of the film occupied by the images of the white zones in Fig. 240 were rendered insoluble by the action of the light, while the rest of the film was unacted upon, and was afterwards dissolved in water. By trial a film was obtained of such thickness that light from the brightest part of the spectrum was retarded an odd number of wave-lengths during transmission through it. Since the film remained over the zones distinguished by odd numbers, the light transmitted through the latter zones suffered the requisite phase change.

Prof. Wood has substituted the above zone plate for the object-glass of a telescope and obtained good definition. The craters on the moon could be seen by its aid. He has also obtained landscape photographs by using a zone plate with a camera.

**Resolving Power of Optical Instruments.**—As we have seen, wave propagation is closely associated with interference. In the new wave surface the wavelets from the old wave surface reinforce each other; at other points the wavelets interfere and produce no resultant effect. We have also seen how a divergent wave, after passing through a convex lens, becomes convergent; the function of the lens is to retard the central portion of the incident wave. The convergent wave thus produced is propagated by reinforcement and interference. The focus is the small space within which all of the secondary wavelets reinforce each other: this space will always possess a certain magnitude, so that the optical image of a geometrical point will never itself be a point, but will possess finite dimensions. The limits of the focus correspond to the points where the wavelets mutually interfere.

FIG. 241.—Resolving Power of a Lens.

Let light-waves diverge from a point, P, on the axis, AP, of a lens, L (Fig. 241), and be brought to a focus in the neighbourhood of A. It is

required to determine the diameter CB of the image formed. The point B is found from the consideration that interference must occur there between the secondary wavelets derived from the main wave after emergence from the lens. It is obvious that the wavelets from E and F, points on the periphery of the lens, are most capable of interference, since their paths differ more than those of other wavelets. Accordingly, (FB − EB) must be equal to $\lambda/2$. Join FC. Then, by symmetry, FC = EB, and FB − FC = $\lambda/2$. Let AL = $v$, while LF = $r$, and CB = $\delta$. Then, by reasoning similar to that used on p. 390, we have—

$$(\lambda/2)\delta = r/v\,; \quad \therefore\ \delta = v\lambda/2r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Between the points C and B, all wavelets reinforce each other, while at C and B interference commences. Thus, CB represents the brightest part of the image; outside the points C and B, the illumination rapidly diminishes.

It thus appears that the diameter of the image is inversely proportional to the aperture of the lens. This is a result of great importance: it explains, for instance, one great advantage possessed by telescopes of large aperture. The stars may be considered merely as geometrical points of light; but the image of a star in a telescope will always possess finite dimensions, which are diminished by increasing the aperture of the telescope. Two stars may be so close together that, when viewed by the aid of a small telescope, their images overlap; on using a telescope of greater aperture, their images may be rendered smaller, so that each is distinct from the other.

In certain optical instruments, spherical aberration is diminished by the use of a stop which allows light to pass only through the central portion of the lens. This arrangement entails a considerable loss of resolving power. Lord Rayleigh has pointed out that it is preferable to use a stop which allows light to pass only through the peripheral portion of the lens; by this means spherical aberration is diminished without any loss of resolving power.

Let G, H, be two luminous points equidistant from, and on opposite sides of, the axis AP (Fig. 241). In order that waves from H and G shall be refracted separately by the lens L, they must arrive at the lens distinct from each other; in other circumstances they will be refracted as one wave through the lens and will form a single image. Thus, if the waves coincide with each other at L, they must be separated by about

half a wave-length at F, or $FG - FH = \lambda/2$. Let P, a point on the axis midway between H and G, be at a distance $u$ from L, while $HG = \delta'$. Then, as before—

$$\lambda/2\delta' = r/u\,; \quad \therefore \ \delta' = u\lambda/2r. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Each of the resultant images, as previously proved, will have a diameter, $\delta$, given by (1). Let D be the distance between the centres of the images. Then, since the magnification of the distance HG by the lens is equal to $v/u$, we have—

$$D/\delta' = v/u\,; \ \therefore \ D = \frac{v}{u} \cdot \frac{u\lambda}{2r} = \frac{v\lambda}{2r} = \delta,$$

so that the images will just touch at their edges, and will thus be just distinguishable.

The above results throw an important light on the theory of the microscope. In the first place they prove, that for high resolving power, a microscope must possess an objective of wide aperture. Further, since the distance, $\delta'$, between the nearest points which can be resolved, is proportional to $\lambda$, it follows that anything which diminishes $\lambda$ increases the resolving power. Since the wave-length of light in a highly refracting medium is smaller than in air, we can understand the advantage of using Abbe's homogeneous immersion (p. 78). Finally, a glance at Fig. 241 shows at once that if HG is sensibly smaller than $\lambda/2$, it would be impossible for waves from H and G to arrive at the lens distinct from each other. Thus, we can never hope to see any object which is *very much* smaller than the wave-length of light ; in particular, we cannot hope ever to see atoms or molecules. This restriction is quite independent of the perfection to which microscopes may be brought : it is inherent in the nature of light. Dr. Woodward, in America, has resolved Nobert's set of test-lines of 112,000 to the inch, *i.e.* about half the length of the blue waves. Much further than this we cannot expect to go.

**Diffraction at a Grating.**—We must now examine the effects produced when a number of narrow opaque obstacles, of equal breadths, distributed at equal intervals in a plane, are interposed in the path of a train of light-waves. An arrangement of the sort described is termed a **grating**. Gratings are usually made by ruling, with a diamond point, fine equidistant lines on the

surface of a sheet of glass. The rulings act as narrow opaque obstacles, separating narrow transparent spaces. Gratings frequently contain as many as 20,000, or even 40,000, lines to the inch; in such cases the lines are invisible except under a powerful microscope. Photographic reproductions of ruled gratings are also frequently used.

Let AB (Fig. 242) represent the section of a plane grating, supposed perpendicular to the plane of the paper. Let the width of the clear space between any two consecutive rulings be equal to $a$, while the width of each ruling is equal to $b$. The distance $(a+b)$, comprising one space and one ruling, will be termed a **grating element**. Points in two consecutive spaces, separated by a distance $(a + b)$, will be termed **corresponding points**. Let a train of plane waves, of any particular period, be incident normally on a grating; the transmitted light may be considered to consist of an indefinitely large number of cylindrical wavelets, each being produced by the disturbance in one of the very narrow strips into which a transparent space may be supposed to be divided. The section of a wavelet by the plane of the paper will be a circle. Remembering that a ray is the path traversed by the disturbance from a particular point in a wavelet, it is readily seen that each wavelet gives rise, in the plane of the paper, to an indefinitely large number of rays diverging from the point at which it takes its origin. Let us now consider the resultant effect produced by the rays derived from all points in the grating spaces; we shall at first confine our attention to rays which make a certain angle, $\theta$, with the normal to the grating. If these parallel rays, as represented in Fig. 242, fall on a lens, L, of which the axis is parallel to the grating normal, they will be brought to a focus at a point, $P_3$, in the focal plane of the lens. Thus, at $P_3$ the illumination is due to the resultant of the disturbances transmitted along the various rays. If the various

FIG. 242.—Diffraction of Plane Waves at a Grating.

disturbances reinforce each other at $P_3$, that point will be brightly illuminated; otherwise it will be dark. We must therefore determine the phases of the disturbances arriving at P after traversing the various rays.

From B, the extremity of one of the grating spaces, draw BC perpendicular to the direction of the rays. A plane wave, of which the section coincides with BC, would, after traversing the lens, converge toward P. In other words, no relative phase change will be produced between the various rays after these pass the line BC. But, before reaching BC, the various rays have traversed different paths; since the disturbances in the plane of the grating are equal in phase, the only phase changes produced are due to the differences in the various paths traversed.

FIG. 243.—Diffraction at a Grating.

Let AB, CD (Fig. 243), represent consecutive grating spaces, separated by the ruling BC. Draw AM perpendicular to the rays which make an angle, $\theta$, with the grating normal. Then AM makes an angle, $\theta$, with the grating surface. Before reaching the line AM, the disturbance from C has traversed the distance CM. On the other hand, the disturbance at A originated there. Thus, the phases of the disturbances at A and M will be equal, or will differ by $\pi$, according as CM is equal to an even or odd number of half wave-lengths. No further phase change will occur in the rays AE and CG before these meet each other in the focal plane of the lens. They will thus reinforce, or interfere with, each other, when brought to a focus by the lens, according as CM is equal to an even or odd number of half wave-lengths. Produce AM to cut the ray DF in K. Then, the phase difference in the rays BH and DF, at the points L and K in the line AK, will be equal to DK − BL = CM. Thus, the rays BH and DF will reinforce, or interfere with, each other, when brought to a focus, according as CM is equal to an even or odd number of half wave-lengths. If we take any two rays, originating at "corresponding" points in the spaces AB and CD, these rays will reinforce, or interfere with, each other, under the same conditions. Consequently, **the whole of the rays from AB will reinforce, or interfere with, the whole of those from CD, according as CM is equal to an even or odd number of**

**half wave-lengths.** Since the grating is supposed to be uniform, the rays in a given direction from pairs of consecutive spaces all over the grating reinforce, or interfere with, each other, according as those from any two consecutive spaces reinforce, or interfere with, each other. Further, $CM = AC \sin CAM = (a + b) \sin \theta$. Thus, the point $P_3$ (Fig. 242) will be brightly illuminated, due to the mutual reinforcement of all rays making an angle $\theta$ with the grating normal, when—

$$(a + b) \sin \theta = n\lambda, \quad \ldots \ldots \ldots \quad (1)$$

where $n$ may have any integral value, including zero. On the other hand, $P_3$ will be dark, when—

$$(a + b) \sin \theta = (n + \tfrac{1}{2})\lambda. \quad \ldots \ldots \quad (2)$$

The point O (Fig. 242) corresponds to zero value of $\theta$. This point will consequently be the centre of a bright band. In passing along OD, we shall arrive at a point for which the corresponding value of $\theta$ satisfies (2), when $n = 0$. Here there will be darkness. Further on, we shall reach a point, $P_1$, for which the corresponding value of $\theta$ satisfies (1), when $n = 1$. This point will be the centre of a bright band. Subsequently, we shall alternately encounter dark and bright bands as we proceed. There will be similar alternations between brightness and darkness as we proceed along OE.

We have previously supposed that the light was monochromatic. It is easily seen that for blue light the point $P_1$ will be closer to O than for red light; this follows from the circumstance that, for $P_1$—

$$\sin \theta_1 = \lambda/(a + b);$$

and the smaller $\lambda$ is, the smaller will be the value of $\theta$ satisfying this equation. Similarly, for blue light the point $P_2$ will be closer to O than for red light. If white light is used, the central image at O will be white, but as we pass along OD we shall successively encounter a number of brilliant spectra, the blue end of each being on the side nearer to O. The spectrum in the neighbourhood of $P_1$ is said to be of the 1st order, while those at $P_2$, $P_3$, . . . are termed the spectra of the 2nd, 3rd, . . . &c. orders. Thus the order of a spectrum is determined by the integral value of $n$ substituted in equation (1) above.

It must be noticed that these diffraction spectra are formed in the focal plane of the lens L. On removing L, they can be seen by the unaided eye when the latter is unaccommodated, so

that parallel rays are focussed on the retina. In that case the optical system of the eye takes the place of the lens L. The diffraction spectra will appear to be at an infinite distance behind the grating.

Diffraction spectra can easily be observed, without special apparatus.

EXPT. 71.—Look at a distant lamp flame through a cambric handkerchief, or through a silk umbrella. A number of coloured images of the flame will be observed, arranged in two rectangular directions, these latter being perpendicular to the meshes of the fabric. The fine scratches formed in cleaning the windows of railway carriages will sometimes produce diffraction spectra of a distant source of light.

It will be seen from equation (1), p. 451, that there is a definite relation between the wave-length, $\lambda$, of the diffracted light: the order, $n$, of the spectrum: the width, $(a+b)$, of a grating element; and the angle of diffraction, $\theta$. Thus, the formation of diffraction spectra by a grating gives us the means of determining the wave-length of light. The most accurate methods of accomplishing this will be described subsequently; the following simple method, which requires no special apparatus, is of some interest.

EXPT. 72.—Coat a piece of plate-glass with tinfoil (p. 395), and cut narrow slits in the latter to form an elongated A (Fig. 244, I). Mount this, with the cross-bar of the A horizontal, in front of a Bunsen flame, into which common salt is introduced (p. 333). Support a piece of fine wire gauze in a plane parallel to that containing the slits, and at some distance in front of the latter. Focus a small telescope on a distant object, and then place it, with its axis perpendicular to the gauze, at a distance of about 3 feet from the latter, and direct toward the centre of the cross-bar of the A.

FIG. 244.—Illustrates Expt. 72.

On looking through the telescope, a direct image of the A is seen, somewhat out of focus, together with several well-focussed diffraction spectra. Adjust the gauze so that one set of wires is vertical, and move it backwards and forwards till the first diffraction spectra intersect each other on the cross-bar of the A (Fig. 244, II). If D is the distance from slit to grating, while $l$ is the length of the cross-bar of the A,

and $\delta$ is equal to the width of a grating element, then the mean wave-length, $\lambda$, of the sodium light is given by—

$$\lambda = l\delta/2D.$$

The values of $l$ and $\delta$ can be obtained by the aid of a travelling microscope, or by forming enlarged images on the screen by the aid of a lens, and determining the magnification.

The theory of the above experiment can be understood by reference to Fig. 245. Let us suppose that A, B, are two parallel slits perpendicular to the plane of the paper, while CE is the grating, and L is the object-glass of the telescope. Rays from A, which pass in straight lines through the grating spaces to the lens, are brought to a focus at A′; the construction for the position of A′ is indicated by the broken lines. A′ is the direct image of A. Similarly, B will give rise to an image B′.

FIG. 245.—Wave-Length Determination, using Two Slits and a Grating.

The diffracted rays which leave the grating normally cross each other at the principal focus, $f$, of the lens L. No relative phase change is produced between these rays after leaving the plane of the grating. Let F, G, H, K be corresponding points of successive grating elements. Join AF, AG, AH, AK. With A as centre, and AF and AG as radii, describe the arcs FM and GN. These arcs are sensibly straight, and perpendicular to AG and AH respectively. Since the width of a grating element is small, AF, AG, AH, &c., will all cut the axis, OX, in points very close to the grating, and MG will be approximately equal to NH. &c. If MG = $\lambda$, the rays diffracted parallel to the axis from F and G will reinforce each other at $f$; similarly, all rays diffracted parallel to the axis, from the grating element FG, and the one below it, will reinforce each other at $f$, and a similar reinforcement will occur with respect to rays diffracted parallel to the axis from other elements. Now, $\angle$ GFM = $\angle$ HGN (approximately), &c. = the angle of incidence, $i$, of the nearly parallel rays AG, AH, &c. Then, if MG = $\lambda$, we have—

$$\lambda/\delta = \sin i = \tan i \text{ (nearly)} = l/2D,$$

where $l/2 = OA$, and D is the perpendicular distance from O to the grating.

$$\therefore \lambda = l\delta/2D.$$

When this relation holds, there will be a bright vertical line at $f$, due to diffracted rays from A. From symmetry, the rays from B, which are diffracted perpendicular to the grating, will also give rise to a bright line at $f$. By moving the grating backwards and forwards until these two diffracted images overlap, we may ensure that the above equation is satisfied. Using two slits inclined to each other, and adjusting so that the inclined diffracted images intersect on the image of the horizontal slit, greater accuracy is obtained.

**Dispersive Power.**—The formula

$$\sin \theta = n\lambda/(a + b)$$

indicates that different values of $\lambda$ correspond to different values of $\theta$. Thus, as already explained, when white light is used as an illuminant, a spectrum is formed somewhat similar in appearance to that produced by a prism. Let $\lambda$ be a particular wave-length corresponding to an angle of diffraction $\theta$ in the spectrum of the $n$th order. In the same spectrum let a slightly longer wave-length, $\lambda + d\lambda$, correspond to a slightly larger angle of diffraction, $\theta + d\theta$. The symbol $d\lambda$, *taken as a whole*, represents a small increase in the wave-length, and the symbol $d\theta$, also taken as a whole, represents the small increase in the corresponding angle of diffraction. Then the **dispersive power** of the grating may be represented by the ratio, $d\theta/d\lambda$. This is equal to the increase in the angle of diffraction corresponding to unit increase in the wave-length.

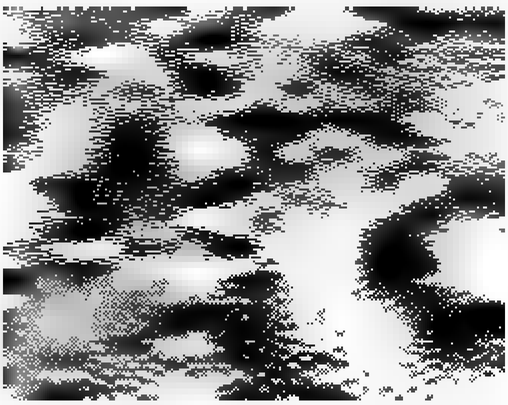

FIG. 246.—To determine the Dispersive Power of a Grating.

Let AB (Fig. 246) represent a grating, comprising the N rulings $a, b, c, d, e, \ldots$ Let C and D be the beginnings of the 1st and (N + 1)th clear spaces, while DF is the direction in which a particular wave-length, $\lambda$, is diffracted to form the bright spectral band of the $n$th order. Draw CE perpendicular to DF, intersecting the latter in E. Then,

since the relative retardation between rays from corresponding points in consecutive spaces amounts to $n\lambda$, and the points C and D are separated by N grating elements, it is obvious that DE = $Nn\lambda$. Let DG be the direction in which the wave-length $\lambda + d\lambda$ is diffracted to form the bright band of the $n$th order. Draw CH perpendicular to DG, intersecting the latter in H. Then, DH = $Nn(\lambda + d\lambda)$. Let the angle of diffraction for the ray DF be $\theta$, while that for the ray DG is $\theta + d\theta$. Then, $\angle$ DCE = $\theta$, and $\angle$ FDG = $d\theta$ = $\angle$ ECH. Also, DE = DC $\sin\theta = N(a + b)\sin\theta$, where $(a + b)$ is the width of a grating element (p. 449). Similarly, CE = DC $\cos\theta = N(a + b)\cos\theta$.

Now, DH is approximately equal to DE, *plus* the circular arc, of radius CE, intercepted between the lines CH (produced) and CE. Thus—

$$\mathrm{DH} = \mathrm{DE} + \mathrm{CE} \cdot d\theta = N(a + b)(\sin\theta + d\theta \cdot \cos\theta).$$

Thus—

$$N(a + b)\sin\theta = Nn\lambda.$$

$$N(a + b)(\sin\theta + d\theta \cdot \cos\theta) = Nn(\lambda + d\lambda).$$

$$\therefore N(a + b) \cdot d\theta \cdot \cos\theta = Nn \cdot d\lambda. \quad . \quad . \quad . \quad . \quad (1)$$

and—

$$\frac{d\theta}{d\lambda} = \frac{n}{(a + b)\cos\theta}.$$

The above result shows that the dispersive power increases with the order of the spectrum observed, and is inversely proportional to the width $(a + b)$ of a grating element. When $\theta$ is small, $\cos\theta = 1$ (approximately), so that in these circumstances equal increments of $\theta$ correspond to approximately equal increments of wave-length. For high order spectra, where $\theta$ is large, the dispersion increases appreciably with $\theta$, and therefore with the wave-length, so that these spectra are more drawn out at the red than at the blue end.

**Resolving Power of Grating.**—We must now determine the breadth of each bright band formed in the focal plane of the lens L (Fig. 242), when the light consists of only one wave-length. This point is one of great importance, since, if the light consists of two wave-lengths which are very nearly equal in magnitude, we shall not be able to obtain two distinct lines in the spectrum unless the breadth of the band due to each wave-length is less than the distance from centre to centre of the two bands.

Let AB (Fig. 246) represent a grating possessing N rulings, $a$, $b$, &c. Let DF be the direction of the rays which reinforce each other at

the centre of the bright band of the $n$th order. Draw CE perpendicular to DF; then $DE = Nn\lambda = N(a + b)\sin\theta$.

Draw the line CH in such a direction that the perpendicular DH, let fall on it from D, is equal to $(Nn + 1)\lambda$. The rays parallel to DG, which proceed from points at opposite edges of the grating, traverse paths differing by $DH = (Nn + 1)\lambda$. Rays parallel to DG, which proceed from points separated by a distance equal to *half* the breadth of the grating, traverse paths which differ by $\{(Nn + 1)\lambda\}/2 = (N/2)n\lambda + \lambda/2$. Let N be even; then the above difference of path amounts to an integral number of wave-lengths, *plus* half a wave-length; consequently the rays interfere and annul each other. Let us now divide the grating into two equal portions by an imaginary line parallel to the rulings. Then a ray parallel to DG, from any point in one half of the grating, will interfere with a parallel ray from a point in the other half of the grating. Thus, the rays from the whole of the first half of the grating will be destroyed by those from the second half. Consequently, light of wave-length equal to $\lambda$, will cease to be diffracted along DG, but will be diffracted along any line lying between DF and DG.

If N, the number of elements, is odd, we may disregard the last element, since the light from a single element can produce no appreciable effect; the above reasoning can then be applied to the remaining elements.

Now draw CK in such a direction that DK, the perpendicular let fall on it from D, is equal to $(Nn - 1)\lambda$. Produce DK to L. Then, by reasoning similar to that used above, it can be proved that light of wave-length $\lambda$ will just cease to be diffracted along the direction DL, but will be diffracted along all lines lying between DL and DF.

It is obvious that the angles GDF and FDL are approximately equal; let each of these angles have the value $d\theta$. Then, since DF makes an angle, $\theta$, with the normal to the grating surface, the $n$th bright band, corresponding to a wave-length $\lambda$, will be formed by rays diffracted at angles lying between $(\theta - d\theta)$ and $(\theta + d\theta)$.

By reasoning similar to that used on p. 455,

$$DE = Nn\lambda = N(a + b)\sin\theta.$$

$$DH = (Nn + 1)\lambda = N(a + b)(\sin\theta + d\theta \,.\, \cos\theta).$$

$$DK = (Nn - 1)\lambda = N(a + b)(\sin\theta - d\theta \,.\, \cos\theta).$$

$$\therefore\ 2N(a + b)\cos\theta \,.\, d\theta = 2\lambda. \quad \ldots\ldots \quad (2)$$

In order that two wave-lengths, $\lambda$ and $\lambda + d\lambda$, shall give separate lines in the $n$th spectrum, their angles of diffraction

must differ by the value of $2d\theta$ given in (2) above. Altering $d\theta$ into $2d\theta$ in (1), p. 455, and substituting in (2) we obtain—

$$Nn \cdot d\lambda = 2\lambda.$$

$$\therefore \frac{d\lambda}{\lambda} = \frac{2}{Nn}. \quad \ldots\ldots\ldots \quad (3)$$

$d\lambda$ is the smallest difference of wave-length, between lines of a mean wave-length equal to $\lambda$, which can be completely resolved in the $n$th spectrum of a grating containing N rulings. It is, of course, assumed that the telescope objective is large enough to take in rays from all the spaces. The ratio $\lambda/d\lambda$ is termed the **resolving power** of the grating. It is obvious that the resolving power increases with the order of the spectrum observed. It is not possible, however, with an ordinary grating, to observe a spectrum of very high order, owing to the decrease in brightness which accompanies an increase in the order. **With a spectrum of given order, the resolving power is proportional to the total number of lines ruled on the grating.**

The D lines differ in wave-length by 6 tenth-metres, and this mean wave-length is (roughly) 6,000 tenth-metres. In order to resolve the D lines in the spectrum of the 2nd order, the grating must possess N lines, where—

$$\frac{2}{N \times 2} = \frac{6}{6{,}000}. \quad \therefore N = 1{,}000.$$

**Absent Spectra.**—It sometimes happens that for a value of $\theta$ satisfying (1), p. 451, no spectrum can be observed. This happens when the value of $\theta$ is such that each space contains an even number of half-period elements. In this case the rays from each space mutually interfere (compare p. 434). For the space AB (Fig. 243) to contain an even number of half-period elements with respect to the direction AE, the distance BL must be equal to some whole number or wave-lengths. For this condition to be satisfied—

$$a \sin\theta = n'\lambda.$$

Also, from (1), p. 451,

$$(a + b)\sin\theta = n\lambda.$$

$$\therefore \frac{a}{a + b} = \frac{n'}{n}.$$

Since $n'$ and $n$ must both be whole numbers, $a/(a + b)$ must be a proper fraction. Let $\alpha$ and $\beta$ be the *smallest whole numbers* which measure the ratio of $a$ to $b$; thus $a = k\alpha$, and $b = k\beta$. Then $\alpha/(\alpha + \beta) = n'/n$. The $(\alpha + \beta)$th, $2(\alpha + \beta)$th, $3(\alpha + \beta)$th, ... &c., spectra will then be wanting, since in these cases the spaces will each contain $2\alpha$, $4\alpha$, $6\alpha$, . . . half-period elements.

When $a = b$, and the spaces and rulings are of equal widths, the 2nd, 4th, 6th, . . . spectra will be wanting. When $a = 2b$, the 3rd, 6th, 9th, . . . spectra will be wanting.

EXPT. 73.—Mount a diffraction grating, with rulings vertical, on the central table of a spectrometer, the telescope and collimator of which have been adjusted as described on p. 88. Illuminate the slit with a sodium flame, and observe the diffraction spectra. The central table should be rotated until the 1st diffraction spectra occur at equal distances on opposite sides of the direct image of the slit. Measure the angular distance between the two first spectra, divide by 2, and substitute the angle so found for $\theta$ in (1), p. 451, where $n = 1$; then calculate the value of $\lambda$. If the grating contains N′ lines per centimetre, $(a + b) = 1/N'$ cm. Obtain values of $\theta$ for the 2nd, 3rd, . . . spectra in a similar manner. Show that $\sin\theta_1 : \sin\theta_2 : \sin\theta_3, \ldots = 1 : 2 : 3, \ldots$

EXPT. 74.—Illuminate the slit with sunlight reflected from a mirror, and obtain the wave-lengths of the principal Fraunhofer lines (p. 340).

EXPT. 75.—Cut a number of rectangular apertures of different widths in cards, and place these, in turn, in front of the grating, and observe the difference produced in the resolving power with respect to the D lines.

**Diffraction in the Microscope.**—When a diffraction grating, or other object possessing a regularly striated structure, is being examined under a microscope, diffraction spectra, similar to those already described, are necessarily formed in the focal plane of the objective. These are not visible through the eye-piece when it is adjusted to give distinct vision of the image of the grating; on removing the eye-piece, and looking down the tube, the diffraction spectra can, however, be seen. Each diffraction spectrum is a region of maximum illumination, due to the mutual reinforcement of wavelets from all points of the object. If we have two gratings, one with twice as many lines to the inch as the other, the spectra due to the former will be twice as widely separated as those due to the latter; in fact, the finer rulings lead to the suppression of the 1st, 3rd, 5th, . . . spectra obtained with the coarser rulings.

The 1st, 2nd, 3rd, . . . spectra due to the finer grating are exactly similar in position and character with the 2nd, 4th, 6th, . . . spectra due to the coarser grating. If, now, by means of a diaphragm with suitable apertures, we stop out the 1st, 3rd, 5th, . . . spectra obtained with the coarser grating, the diffraction effects are similar to those obtained with the finer grating. On now replacing the eye-piece, adjusted to view the image of the grating, it is found that this image comprises *twice as many lines* as are actually possessed by the grating; in fact, the appearance presented is identical with that of the finer grating mentioned above. By stopping out every second spectrum still remaining, the number of lines seen in the image is again doubled. On stopping out all spectra except the central (direct) one, no lines at all are seen in the image; the image now formed is similar to that of a grating too finely divided to be resolved by the objective. It thus appears that, in order that a microscope shall be able to resolve an object possessing a regularly striated structure, it is necessary that *at least* the two first diffraction spectra (on opposite sides of the central one) should be visible on looking down the tube after removing the eye-piece.

**Concave Reflecting Gratings.**—If a polished surface is ruled with fine equidistant lines, diffraction effects can be produced by reflecting light from the polished strips between the rulings. Reflection gratings are generally made by ruling the surface of a concave mirror of polished speculum metal with lines which lie in parallel equidistant planes. In this case, as we shall see, no lens is needed, the spectra being focussed by the mirror itself.

FIG. 247.—Diffraction at a Concave Reflecting Grating.

Let A, C (Fig. 247), be corresponding points in consecutive polished spaces of a grating (similar, for example, to A and C in Fig. 243), ruled on a concave surface with centre at K. Let light radiate from a slit, S, perpendicular to the paper. It is required to determine whether the diffracted rays AO and CO reinforce, or interfere with, each other at their point of intersection.

Draw the radius KA. Then, $\angle$ SAK is the angle of incidence, $i$, of the ray SA. Similarly, $\angle$ KAO is the angle of diffraction, $\theta$, of the ray AO. Since the points C and A are very close together, the angles of incidence and diffraction of the rays SC and CO are respectively equal to $i$ and $\theta$, to a close approximation.

With O as centre, and radius OC, describe the arc CD. With S as centre, and radius SA, describe the arc AE. The lines CD and AE are sensibly straight, and respectively perpendicular to AO and SC.

Since AE is also perpendicular to AS, and AC is perpendicular to AK, $\angle$ CAE $= \angle$ SAK $= i$. Similarly, $\angle$ ACD $= \angle$ KAO $= \theta$. Since the phase change produced by reflection at A will be equal to that produced by reflection at C, the difference in phase of the wave disturbances arriving at O will be due merely to the difference in the paths SCO and SAO. Also, since SE = SA, and OC = OD, the difference in the paths SCO and SAO is equal to EC − DA. If AC $= (a + b)$ (compare p. 449), then EC − DA = AC {sin CAE − sin ACD} $= (a + b)(\sin i - \sin\theta)$. Thus, the rays AO and CO will reinforce each other at O when—

$$(a + b)(\sin i - \sin\theta) = n\lambda,$$

where $n$ is any integer; $n$ will have negative values when $\theta > i$.

Let NML (Fig. 248) be a concave surface with centre of curvature at K. Let this be ruled with lines formed by the intersections of the surface with parallel equidistant planes, perpendicular to the plane of the paper, one of these planes passing through the radius, KM, drawn from K to the middle point, M, of the surface. On KM as diameter describe the circle KSMO. Let an illuminated slit, perpendicular to the plane of the paper, be situated at S, a point on the circle KSMO, and let AC, AC′, be any two grating elements. We must determine the illumination at O,

FIG. 248.—Illustrates the Theory of the Concave Reflecting Grating.

a point on the circle KSMO, due to the diffracted rays AO, CO, A′O, C′O, from the grating elements AC and A′C′. Let the angles of incidence and diffraction at A be respectively equal to $i$ and $\theta$, while the corresponding angles at A′ are equal to $i'$ and $\theta'$. Then, if AC = A′C′ = $(a + b)$, the relative retardation between the waves arriving at O along the paths SCO and SAO is equal to $(a + b)(\sin i - \sin\theta)$. Similarly, the relative retardation between the waves arriving at O along the paths SC′O and SA′O is equal to $(a + b)(\sin i' - \sin\theta')$. It will now be proved that, whatever may be the position of A′C′, $i' = i$, and $\theta' = \theta$.

Join KA, KA′. Then, since K is the centre of curvature of NML, ∠ SAK = $i$, and ∠ KAO = $\theta$. Similarly, ∠ SA′K = $i'$, and KA′O = $\theta'$. If the diameter of the mirror is small in comparison with its radius of curvature, the points A, C, A′, C′, will lie very close to the circle KSMO, and, as far as the angles $i$, $i'$, $\theta$, $\theta'$ are concerned, may be assumed to lie on that circle. Then, since the angles SAK and SA′K are subtended, by the same arc SK, at points A and A′ on the circumference of the circle KSMO, these angles are equal, or $i = i'$. Similarly, since the arc KO subtends the angles KAO and KA′O at points A and A′ on the circumference of the circle KSMO, these angles are equal, or $\theta = \theta'$.

Since A′C′ may be any grating element whatever, it follows that if diffracted rays from any two consecutive grating spaces reinforce each other at O, those from all pairs of grating spaces will do so. Proceeding from K along KOM, $\theta$ may be caused to vary between 0 and $\pi/2$. A number of points can be found along KOM, such that the corresponding values of $\theta$ satisfy the equation—

$$(a + b)(\sin i - \sin\theta) = n\lambda,$$

where $n$ has the values 0, 1, 2, 3, . . .

Thus if a slit, S, illuminated by monochromatic light, is situated on the circumference of the circle KSMO, a number of well focussed images of the slit will be situated on the circumference of the same circle. If white light is used as an illuminant, a number of pure spectra will be formed round the circle KSMO.

**Rowland's Grating.**—The late Prof. Rowland was the first to succeed in ruling fine gratings on concave speculum metal mirrors. He utilised the principles explained above, in a most masterly manner, so as to obtain a **normal solar spectrum** ; *i.e.*

a spectrum in which equal distances correspond *exactly* to equal increments of wave-length.

Two rails, SA, SB (Fig. 249), are mounted on a strong framework, so that their directions intersect perpendicularly at S. GC is a wrought-iron girder pivoted near its ends, directly over the rails, on carriages which run along the latter. Then GSC is a right-angled triangle, and in all positions which the carriages may occupy S will be on the circumference of the circle of which CG is the diameter. A vertical slit is placed at S, the intersection of the rails, and a concave grating, with its lines vertical, is mounted at G on the girder CG. The radius of curvature of the grating is equal to CG, and the axis of the grating is adjusted to be parallel to the length of the girder. Monochromatic rays diffracted so as to cross each other at C, will have $\theta = 0$, and $i = \angle$ SGC. Therefore, in order that a wave-length, $\lambda$, should produce a bright line at C—

FIG. 249.—Rowland's Grating and its Accessories.

$$(a + b)\sin i = (a + b)\mathrm{SC/CG} = n\lambda.$$

Since CG is constant, it follows that as the point C moves from S along SB, the 1st, 2nd, 3rd, . . . diffraction images of the slit are encountered, separated by exactly equal intervals. If white light is used to illuminate the slit, the spectra of the 1st, 2nd, 3rd, . . . orders will be encountered as we pass along SB. These spectra lie along the circle of which CG is the diameter, but for a spectrum of any particular order the wave-length $\lambda$, which forms a bright line at C, is exactly proportional to the distance SC. Thus, the rail SC can be graduated in wave-lengths. An eye-piece can be used to observe the spectra, or the latter may be photographed directly by allowing the light to fall, at C, on a prepared photographic plate bent into a short arc of the circle CSG.

Rowland used a slit about 0·025 mm. wide. A grating containing 10,000 lines to the inch was generally used; a space $5\frac{1}{2}$ inches wide was ruled on a 6-inch polished surface, of which the radius of curvature was about 21·5 feet. The photographic plates were about 20 inches long, 2 inches wide, and $\frac{1}{16}$ inch thick. The following extract describes Rowland's method of photographing the solar spectrum :—

"We put in the sensitive plate, . . . and move to the part we wish to photograph. Having exposed that part, we move to another position and expose once more. We have no thought for the focus, for that remains perfect, but simply refer to the table giving the proper exposure for that part of the spectrum, and so have a perfect plate. Thus, we can

FIG. 250.—Portion of Rowland's Solar Spectrum, from $\lambda = 3900$ to $\lambda = 4000$, (much reduced). The two broad bands are the H and K lines.

photograph the whole spectrum . . . in a few minutes, from the F line to the extreme violet, in several strips, each 20 inches long (Fig. 250), and we may photograph to the red rays by prolonged exposure. Thus, the work of days with any other apparatus becomes the work of hours with this. Furthermore, each plate is to scale, an inch on any one of the strips representing *exactly* so much difference of wave-length."[1]

**Overlapping of Spectra.**—When white light is diffracted at a grating, the wave-lengths $\lambda, \lambda_1, \lambda_2, \ldots$ which are diffracted in a direction making an angle $\theta$ with the normal, are given by the equation—

$$(a + b) \sin \theta = n\lambda = (n-1)\lambda_1 = (n-2)\lambda_2 = (n-3)\lambda_3 = \ldots = 1 \times \lambda_n,$$

where $\lambda$ is supposed to be a violet line in the $n$th spectrum, and $\lambda_1, \lambda_2 \ldots$ are longer wave-lengths corresponding to spectra of lower orders. Thus—

$$\lambda_1 = n\lambda/(n-1), \quad \lambda_2 = n\lambda/(n-2), \ldots$$

The visible spectra of the 1st and 2nd orders do not overlap each other; the shortest wave-length in the 1st order spectrum which would overlap the violet line $\lambda$ in the 2nd order

[1] H. A. Rowland, *Phil. Mag.*, p. 197, September, 1883.

spectrum is given by $\lambda_1 = 2\lambda$, and since the visible spectrum comprises slightly less than an octave, $\lambda_1$ would be an infra-red wave-length. The line $\lambda_1 = 3\lambda/2$ in the 2nd order spectrum overlaps the line $\lambda$ in the 3rd order spectrum; if $\lambda$ is a line in the violet (say 4,000), then $\lambda_1$ will be a line in the orange (6,000), so that the orange of the 2nd order spectrum overlaps the violet of the 3rd order spectrum. This overlapping becomes more and more noticeable as the order of spectra increases.

If we examine the spectra, not only for visible radiations, but also for the infra-red and ultra-violet rays, no spectrum will be free from overlapping. The infra-red wave-lengths from the 1st order spectrum, which lie between 8,000 and 16,000 tenth-metres, will bodily overlap the visible spectrum of the 2nd order. Thus, it is practically impossible to use a diffraction grating to analyse the infra-red solar spectrum. For this reason Langley used prisms in his classical researches (p. 344). We may, however, determine the wave-length of any particular part of the infra-red spectrum, if we can suppress all wave-lengths other than that which we wish to measure.

**Wave-length of Infra-Red Rays.**—Langley used a concave reflecting grating to calibrate his infra-red prismatic spectrum, after the following manner. Three arms, each equal in length to half the radius of curvature of the grating, were pivoted at D (Fig. 248). One arm carried the grating NL, a second arm carried a bolometer, while the third arm carried a screen pierced with a narrow vertical slit. The invisible spectrum was caused to traverse the screen by rotating the prism; the radiations at any instant falling on the slit were diffracted by the grating NL, and as the arm carrying the bolometer was rotated, the bolometer traversed the circle KOM, and a deflection of the galvanometer occurred at each point where the diffracted waves reinforced each other. The wave-length was then calculated from the formula given on p. 461.

To determine the wave-length of the residual rays after repeated reflections from rock-salt, sylvine, &c. (p. 384), Rubens used the arrangement represented in Fig. 251. Radiations from a heated Welsbach mantle, A, fell on a slit, $s_1$, and were rendered parallel by reflection at a concave mirror, $e_1$. They then traversed a grating, $g$, made from silver wires, each 0·1858 mm. in diameter, arranged parallel to each other at intervals of 0·1858 mm. The diffracted radiations fell

on a concave mirror, $e_2$, by means of which they were brought to a focus on a slit, $s_2$, in a fixed screen. The mantle A, the slit $s_1$, the mirror $e_1$, and the grating $g$, were supported on a framework which could be rotated about a vertical axis through $g$, so as to bring point after point of the diffracted spectra on $s_2$. After traversing $s_2$, the radiations were reflected from the blocks of rock-salt or sylvine, $P_1$, $P_2$, $P_3$, $P_4$, and then fell on another concave mirror, S. By this latter they were caused

FIG. 251.—Apparatus for determining the Wave-length of the Residual Rays reflected from Rock-Salt or Sylvine.

to converge, so that, after reflection at one more block of rock-salt or sylvine, $P_5$, they were brought to a focus on a delicate thermo-electric pile, nearly surrounded by a metallic vessel, T. The rays not at first absorbed by the pile were reflected back to it from the walls of the vessel, thus increasing the delicacy of the pile. The pile comprised twenty elements, arranged in a line 18 mms. long. The terminals of the pile were connected with a galvanometer so sensitive that an elevation of a millionth of a degree Centigrade produced a deflection of 1 mm.

The mantle A, the slit $s_1$, the mirror $e_1$, and the grating $g$, were at first adjusted so that the central (direct) image was thrown on the slit $s_2$. A deflection of the galvanometer was then observed. On rotating the framework carrying A, $s_1$, $e_1$, and $g$, this deflection rapidly diminished, but, after a certain rotation, was succeeded by another deflection, corresponding to the first diffracted image formed by the residual rays on the pile. The wave-length was then calculated from the angles of incidence and diffraction of the radiations, and the known width of a grating interval. A screen, $k$, could be interposed to cut off the radiations from the grating.

**Difficulties in Ruling a Grating.**—The resolving power of a grating is, as we have seen, proportional to the product of the order of the spectrum observed, and the total number of lines in the grating. With a ruled grating the spectra of high orders are too faint to be utilised experimentally; hence, in order to increase the resolving power of gratings, the general practice has been to rule as many lines as possible. Of course the resolving power can only be increased in this manner when the rulings are exactly similar, parallel, and equidistant, throughout the whole of the grating space.

To rule a fine grating with a great number of equidistant lines is a matter of great difficulty. An automatic dividing engine, driven by clock-work, is used; the ruling diamond point is advanced through a definite distance between each two rulings, by the aid of a carefully cut screw. Any imperfection in the screw will, of course, lead to irregularities in the grating; and when a sufficiently accurate screw has been made, much time may be lost in selecting a suitable diamond point, and the latter may break down before a grating is finished. Under favourable conditions it takes five days and nights to rule a 6-inch grating having 20,000 lines to the inch; and during the whole of that time the temperature of the ruling engine must be kept constant to within a fraction of a degree Centigrade, or the expansion of the screw will lead to irregular ruling. Thus, it becomes apparent, that the resolving power obtainable by means of a ruled grating is limited by mechanical difficulties which it will always be hard to overcome. Few ruled gratings possess a resolving power as high as that calculated from (3), p. 457.

**Michelson's Echelon Grating.**—Professor Michelson has invented an ingenious arrangement by which an enormous

increase in resolving power may be obtained. His object was to construct a grating with which spectra of very high orders might be observed. To secure this result, a number of exactly similar plates of glass are built up *in echelon* (or in steps) (Fig. 252), all steps being equal in width. Parallel light falls normally on the largest plate, and traverses the system as indicated by the shading in Fig. 252. The light emerging at each step may be decomposed into wavelets, and each wavelet is equivalent to a number of radiating rays; the rays from consecutive steps, which reinforce each other in any particular direction, form a bright band in the focal plane of a lens placed in front of the "echelon."

FIG. 252.—Michelson's Echelon Grating.

Let AB, CD (Fig. 253), be two consecutive steps of an echelon grating. Let AB = CD = $a$, while the thickness, BC, of each plate is equal to $t$. Light arrives at all points of AB in the same phase. Light also arrives at all points of CD in the same phase, but this phase generally differs from that corresponding to AB, since, after passing the plane AB, the light has to traverse a thickness $t$ of glass before reaching CD. Let $\mu$ be the refractive index of the glass for waves of length $\lambda$. Then, for these waves, a distance $t$ in glass is equivalent to a distance $\mu t$ in air.

FIG. 253.—Theory of the Echelon Grating.

Let AE, CF, be parallel rays from corresponding points in the steps AB and CD. Draw CG perpendicular to these rays. Then, before reaching the plane of which CG is the trace, the ray from C has traversed the distance BC in glass, while that from A has traversed the distance AG in air. Draw BH parallel to AE, and draw AK perpendicular to BH. Then, if $\angle$ CBH = $\theta$, AG = KL = BL − BK = BC cos CBL − AB cos ABK

$= t\cos\theta - a\sin\theta$. Thus, in order that the rays AE and CF shall be in the same phase on reaching the plane GC, the difference in the paths of these rays, after passing through the plane AB, must be equal to an integral number of wave-lengths; or—

$$\mu t - (t\cos\theta - a\sin\theta) = n\lambda.$$

When the rays AE and CF are in the same phase in the plane GC, any two rays parallel to AE and CF, from corresponding points in the steps AB and CD, will be in the same phase in the plane GC. If all rays from the steps fall on a lens, those which make an angle $\theta$ with BC will reinforce each other at a point in the focal plane of the lens. Thus, the above equation gives the condition for a bright band to be formed by the rays diffracted from all the steps (Fig. 252) at an angle $\theta$.

When $\theta$ is small, $\cos\theta = 1$, and $\sin\theta = \theta$ (approximately). In this case we have—

$$(\mu - 1)t + a\theta = n\lambda. \quad \ldots\ldots\ldots \quad (1)$$

Echelon gratings have been made with thirty steps, each glass plate being 7·8 mms. in thickness, the width ($a$) of a step being equal to 1 mm. To determine the general order of the spectra formed, put $\theta = 0$ in (1), while $\lambda = 0·6 \times 10^{-3}$ mm. (roughly the wave-length of the D lines). Then, if $\mu = 1·6$—

$$n = \frac{0·6 \times 7·8}{0·6 \times 10^{-3}} = 7·8 \times 10^3 = 8{,}000 \text{ (roughly).}$$

Thus, with this instrument, the spectra observed are of the 8,000th order. The resolving power of the instrument (p. 457) is roughly equal to—

$$\frac{\lambda}{d\lambda} = \frac{30 \times 8{,}000}{2} = 120{,}000.$$

$$\therefore\ d\lambda = \lambda/1·2 \times 10^5.$$

Writing $\lambda = 0·6 \times 10^{-3}$ mm., we find that wave-lengths differing by $d\lambda$ could be resolved, where—

$$d\lambda = 0·6 \times 10^{-3}/(1·2 \times 10^5) = 5 \times 10^{-9}\,\text{mm.}$$

Remembering that the D lines differ by 6 tenth-metres, or $6 \times 10^{-7}$ mm., we see that the echelon grating is capable of resolving lines differing by less than a hundredth part of the difference between the D lines. Such a result could scarcely be attained by the best ruled grating procurable, owing to unavoidable irregularities in the ruling.

The echelon grating is of no use for the analysis of sunlight and similar purposes. The high order of the spectra observed

makes overlapping inevitable. It is only used for examining particular lines to see whether they are doublets, triplets, &c. The collimator and telescope of a spectrometer are adjusted in the usual manner, and the echelon is mounted on the central table, the end plate being perpendicular to the axis of the collimator. Light from a vacuum tube is analysed by a prism, and one of the lines is thrown on the slit of the echelon spectrometer. The diffracted images are then observed by the telescope. In this manner the red hydrogen line, 6563, has been observed to be double, as predicted by Michelson (p. 421); the difference in wave-length between the components amounts to $1 \cdot 4 \times 10^{-8}$ mm.

The plates used in making an echelon grating must be exactly equal in thickness, and their faces must be truly parallel. To secure this result, they are cut from a single plate, the faces of the latter having been ground parallel. Before being cut, the plate is placed in the path of one of the interfering pencils of a Michelson's Interferometer (p. 418); any irregularity in the surfaces of the plate produces a distortion of the interference fringes.

## Questions on Chapter XVII

1. Light from a luminous point casts a shadow of a straight edge. Describe and explain the fringes seen near the edge of the shadow.

2. Light is diffracted by a narrow rectangular obstacle: show how to determine the position of maxima and minima of brightness in the neighbourhood of the edge of the geometrical shadow.

3. The photograph of the shadow of a very small circular disc, when enlarged, is found to have a bright spot in the centre. Explain this.

4. Light issuing through a narrow vertical slit is received by a lens after passing a fine vertical wire. What appearances will be presented, and how may they be explained and demonstrated?

5. Homogeneous light from a point-source falls upon a convex lens. Show, by general reasoning, that at the geometrical focus there will be a central bright spot surrounded by dark and bright rings, and show that the area of the central bright spot will be smaller the greater the aperture of the lens,

6. Describe the method of producing a diffraction spectrum. Explain what is meant by diffraction spectra of different orders, and state by what considerations you would be guided in selecting for observation the spectrum of a particular order.

7. Explain the action of a diffraction grating, and prove the formula which connects the wave-length of the diffracted light, its deviation, and the distance between the lines of the grating.

8. A beam of white light proceeding from a narrow slit is thrown, by means of a lens, on to a screen. Explain the appearance which would be produced by placing a diffraction grating between the lens and the screen, in the path of the beam.

9. Give an account of the method of using a diffraction grating, and explain how the spectrum produced differs from a prismatic spectrum.

10. Describe fully the method of determining the wave-length of light by means of the diffraction spectrum.

11. Describe, and give the theory of, the method of determining the wave-length of light, by means of two slits and a diffraction grating.

12. Fine parallel lines are ruled closely at regular intervals on a glass plate; explain precisely why and where spectra are formed when the plate is illumined by a beam of light coming through a lens from a slit arranged parallel to the ruled lines. Explain the effect on these spectra, of the closeness of the ruling, and the breadth of the ruled space.

13. A diffraction grating is formed on a spherical surface, and a slit —parallel to the lines of the grating—is placed at the centre of the sphere. Show that the spectra formed lie on a circle passing through the centre of the grating, and the centre of the sphere. How is this arrangement realised in practise so as to photograph the spectrum?

14. How is the resolving power of a diffraction grating defined, and how may it be calculated and tested in any given case?

## PRACTICAL

1. Determine the wave-length of the red lithium line by means of the given diffraction grating.

2. Determine the number of lines per centimetre in the ruling of the given diffraction grating.

## CHAPTER XVIII

### POLARISATION AND DOUBLE REFRACTION

**Polarisation by Reflection.**—We have seen that, after being transmitted through a tourmaline crystal, light exhibits an absence of symmetry about the axis of the ray (p. 324); it can be transmitted by a second tourmaline crystal if the axes of the two crystals are parallel, but it is intercepted by the second crystal when the axes are perpendicular to each other. This result leads to the conclusion that light consists of waves in which the displacements are transverse to the direction of propagation. After traversing the first tourmaline crystal, the displacement in the transmitted waves is confined to a single direction, and the light is said to be **polarised.** Thus, tourmaline possesses the property of transmitting waves in which the displacements are parallel to a certain direction, and intercepting waves in which the displacements are perpendicular to that direction.

Light may also be polarised more or less completely by reflection, at an appropriate angle, from the surface of a transparent medium. We can, of course, detect the presence of polarisation in the reflected light by allowing it to traverse a tourmaline crystal before reaching the eye: if a rotation of the tourmaline, about the ray as axis, produces any change in the brightness of the transmitted light, partial polarisation has been produced; if the tourmaline in any position extinguishes the light, the latter must have been completely polarised.

EXPT. 76.—**Observe the light, reflected from the surface of water, glass, glazed earthenware, &c., through a single tourmaline crystal, as**

the latter is rotated about the line of vision. More or less variation in the brightness of the transmitted light occurs as the crystal is rotated.

When light is reflected from a plane surface, the incident and reflected rays, together with the normal to the surface, lie in a single plane, termed the **Plane of Incidence.** The angle which the incident ray makes with the normal to the surface is termed the **Angle of Incidence.** Now, light-waves in which the displacements are *perpendicular* to the plane of incidence, will be reflected under different conditions from those in which the displacements are *in* the plane of incidence. In the first case, the wave displacements, being perpendicular to the plane of incidence, must be perpendicular to the normal to the surface, and therefore *parallel to the surface*, whatever the angle of incidence may be. In the second case, the wave displacements, being perpendicular to the incident ray and in the plane of incidence, will make *various angles with the surface* as the angle of incidence is varied. There is, then, a certain amount of *a priori* probability that light-waves, in which the displacements are perpendicular to the plane of incidence, will be reflected for all angles of incidence; while there may be a certain angle of incidence for which the waves, comprising displacements only in the plane of incidence, will be transmitted without giving rise to a reflected ray. If these conjectures are correct, we should find that, for a certain angle of incidence, no light can be reflected except that in which the wave displacements are perpendicular to the plane of incidence; while for the same angle of incidence the transmitted light contains all incident waves in which the displacements are in the plane of incidence, together with a certain proportion of the waves in which the displacements are perpendicular to the plane of incidence. Thus, the reflected light would be completely polarised, while the transmitted light would be partially polarised.

Examination of the light reflected from a sheet of black glass, shows that for a certain angle of incidence, the reflected rays are almost entirely extinguished by a tourmaline crystal with its axis in a certain direction; in other words, the light reflected from glass, at a particular angle, is polarised. The object of using black glass is merely to absorb the rays which, in the case of clear glass, would be internally reflected from the second surface. The light is **polarised by reflection** at the surface of the glass; the polarisation is detected, or the reflected light is **analysed**, by means of the tourmaline crystal. But a second sheet of black glass may be used to analyse the reflected

light. If we arrange two sheets of glass so that both are equally inclined to the horizon, then, if a ray, reflected vertically from the first plate, falls on the second plate, the angles of incidence in both cases are equal. Keeping the inclinations of both plates to the horizon constant, we can rotate the second plate about a vertical axis, until its normal lies in a plane perpendicular to the plane of incidence with respect to the first plate. In this case, supposing the displacements in the ray reflected from the first plate to be perpendicular to the plane of incidence, these displacements will be *in* the plane of incidence of the ray on the second plate, and so will not be reflected from the latter. Fig. 254 represents a piece of apparatus by means of which these conclusions may be tested. $A_1$ is a glass mirror which can be rotated about a horizontal axis. Above this is placed a black glass mirror, $A_2$, which can also be rotated about a horizontal axis. The supports carrying $A_2$ are fixed to an annulus of metal which can rotate about a vertical axis, over a circular scale graduated in degrees. When $A_1$ and $A_2$ are equally inclined to the horizon, the light reflected from $A_2$ varies in brightness as the framework carrying $A_2$ is rotated about a vertical axis. The brightness is greatest when the normals to $A_1$ and $A_2$ lie in the same plane; when the normals to $A_1$ and $A_2$ lie in perpendicular planes, the brightness of the reflected light is least. The variation which occurs in the brightness of the light reflected from $A_2$ is found to depend on the angle of

FIG. 254.—Biot's Polariscope.

incidence. If the latter is equal to $57\frac{1}{2}^{\circ}$, no light is reflected from $A_2$ when the normals to $A_1$ and $A_2$ are in vertical planes perpendicular to each other; thus, for this angle of incidence, light is almost completely polarised by reflection at a glass surface.

The above experiment proves that **when light is incident at an angle of $57\frac{1}{2}^{\circ}$ on a glass surface, the reflected light consists of waves in which the displacements are confined to a certain definite direction at right angles to the ray.** No information is obtained, however, as to whether the displacements are in, or perpendicular to, the plane of incidence: this point must be investigated by other methods. In order to avoid any assumption in this respect, we may say that **the light reflected from glass at $57\frac{1}{2}^{\circ}$ is polarised in the plane of incidence.** The transmitted light comprises that part of the incident light which is not reflected from the surface, and thus has a deficiency of waves polarised in the plane of incidence, or an excess of waves polarised perpendicular to the plane of incidence—assuming that the incident light consists of numerous waves in which the transverse displacements make all possible angles with the plane of incidence.

**Plane of Polarisation.**—Let a ray of light, polarised by transmission through a tourmaline crystal, be incident on a plate of black glass at an angle of $57\frac{1}{2}^{\circ}$. As the tourmaline is rotated through four right angles about the ray as axis, the light reflected from the glass twice acquires maximum brightness and is twice extinguished. When the light transmitted by the tourmaline is most plentifully reflected from the glass, it must be polarised in the plane of incidence. We thus determine the plane in which the light transmitted by the tourmaline is polarised; this plane is termed the **Plane of Polarisation.** Experiment shows that the plane of polarisation of tourmaline is perpendicular to the axis of the crystal.

FIG. 255.—Illustrates Brewster's Law.

**Brewster's Law.**—The polarisation of light by reflection from glass was discovered by Malus in 1808. About 1811, Sir David Brewster commenced a series of experiments with regard to the polarisation of light by reflection

at the surfaces of various media. He found that light is almost completely polarised in the plane of incidence when reflected from a transparent medium at a particular angle, termed the **Angle of Polarisation.** This angle varies from one medium to another. His researches led to the remarkable law that **the tangent of the angle of polarisation is numerically equal to the index of refraction of the reflecting medium.** Thus, in order that light should be polarised by reflection at the surface of a medium of refractive index $\mu$, the angle of incidence, $i$, must satisfy the equation—

$$\tan i = \mu.$$

In this case it can easily be shown that **the reflected and refracted rays are at right angles.** For, since $\sin i/\sin r = \mu$, we have—

$$\tan i = \sin i/\cos i = \mu = \sin i/\sin r.$$

$$\therefore \cos i = \sin r = \cos\left(\frac{\pi}{2} - r\right).$$

$$\therefore i = \frac{\pi}{2} - r, \text{ and } i + r = 90°.$$

A glance at Fig. 255 will show that this relation can only be satisfied when the reflected and refracted rays are at right angles.

Brewster's Law applies to reflection, not only at a denser, but also at a rarer, medium. It can now be proved that **if light is incident at the polarising angle on the upper surface of a sheet of glass, the refracted ray will be incident at the polarising angle on the lower surface of the glass.** Let $i$ and $r$ be the angles of incidence and refraction at the upper surface; then $r$ and $i$ will be the angles of incidence and refraction at the lower surface, and—

$$\sin r/\sin i = \frac{1}{\mu},$$

while—

$$\tan r = \frac{\sin r}{\cos r} = \frac{\sin i}{\cos r}\frac{1}{\mu} = \frac{1}{\mu},$$

since $i$ is the angle of polarisation at the first surface, and therefore $\sin i = \cos r$.

Thus, the light reflected from both the upper and lower surfaces of a sheet of glass will be polarised in the plane of incidence, if the light is incident on the upper surface at the polarising angle.

Since the refractive index of a substance varies with the wave-length of the incident light, it follows that a substance will possess different

angles of polarisation with respect to the different components of white light, and polarisation can only be complete for one particular wave-length at a time.

M. Jamin has found that, in practice, only a few substances, of refractive index about 1·46, *completely* polarise light by reflection. As the angle of incidence of the light increases, from 0 to $\pi/2$, the proportion of polarised light in the reflected ray at first increases, then reaches a maximum, and finally decreases. An experiment performed by Lord Rayleigh appears to explain this. In ordinary circumstances light is not completely polarised by reflection from water. It is known, however, from surface tension experiments, that the surface of water is very easily contaminated. To determine whether this surface contamination affects the degree of polarisation at the polarising angle, Lord Rayleigh caused a stream of air to blow along the surface of water contained in a long metal trough, so that all impurities were blown up to one end, and then confined there by a partition let down into the water. It was then found that **light of any particular wave-length is completely polarised by reflection at the uncontaminated water surface,** the angle of polarisation being that given by Brewster's Law. Thus, it is probable that the incomplete polarisation of light when reflected from the surfaces of many substances at the angles given by Brewster's Law, is due to unavoidable imperfections in the polish of the surfaces. Sir G. Conroy has found that the optical properties of a glass surface change rapidly during the first few days after the final polishing.

**Pile of Plates.**—Only a small fraction of the incident light is reflected from the surface of glass, or any similar transparent medium, so that the light reflected at the polarising angle is very faint. This defect may be overcome by reflecting light at the polarising angle from a pile of glass plates laid one upon another. At each succeeding plate the reflected light is enriched in waves polarised in the plane of incidence, and the transmitted light is left with fewer of these waves, so that, with a reasonable number of plates, the reflected light contains practically all of the incident waves polarised in the plane of incidence, and the transmitted light contains the rest of the waves, which are polarised perpendicularly to the plane of incidence. The curves in Fig. 256, which have been drawn from data published by Sir G. Stokes, show the percentage of the incident light, polarised in the plane of incidence, which is reflected, and also of that which is transmitted, by piles comprising different numbers of plates. It will be seen that 90 per cent. of the

incident light polarised in the plane of incidence is reflected from a pile of about 24 plates, the remaining 10 per cent. being transmitted; a further increase in the number of plates does not entail a proportional advantage.

Thus, a pile of glass plates affords us a ready and fairly efficient means of polarising or analysing light. Lanterns for optical projection are frequently fitted with an elbow tube in which the light is polarised by reflection at a pile of glass plates.

FIG. 256.—Percentage of Light, polarised in the Plane of Incidence, reflected and transmitted by Piles of Plates.

EXPT. 77.—Obtain a number of microscope cover-slips, and support these within a cardboard tube of suitable size, so that they are inclined at an angle of about 33° to the axis of the tube. A cork bored with a hole of about $\frac{1}{2}$ inch diameter, and then cut with a sharp knife at an angle of 33° with the axis, may be used to clamp the cover-slips in position. About 16 to 24 cover-slips should be used. Light travelling in the direction of the axis of the tube will fall on the plates at the polarising angle, and the transmitted light will be polarised perpendicularly to the plane of incidence. With this simple appliance many interesting experiments can be performed.

**Absolute Direction of Displacement in Polarised Light.**—We have seen (p. 472) that with regard to transverse waves reflected from a transparent medium, we might reasonably expect a variation in the angle of incidence to produce the greatest effect when the wave displacements are in the plane of incidence. Hence, we anticipated that light polarised in the plane of incidence (which is reflected at all angles from a transparent surface) consists of waves in which the displacements are perpendicular to the plane of incidence, and thus always parallel to the surface. On the other hand, the waves which are entirely transmitted at the polarising angle may be conjectured to comprise only displacements in the plane of incidence. It is not safe, however, to lay much stress on such general con-

siderations; as we shall see in the next chapter that the results to be anticipated from strict reasoning depend on the mechanical theory of wave propagation which we adopt. At present we shall merely consider the experimental evidence as to the direction of displacement in polarised light. The earliest evidence in this connection was obtained by Sir G. Stokes, from experiments on the diffraction of polarised light. As his conclusions agree with those arrived at by later experimenters, who used methods more easily explained, and less subject to error from mechanical causes, these later methods alone will be described.

**Scattering of Light by Small Particles.**—As we have seen (p. 376), ether waves tend to set the ultimate constituent particles of matter in a state of vibration. When the period of the waves is very long, there is no appreciable relative displacement between the ether and the material particles; the latter follow the wave displacements just as a cork rises and falls on the smooth rolling waves of the open sea. With waves of very quick periods the case is different. There is here very little absolute displacement of the material particles; before an impulse, let us say from the crest of a wave, has had time to appreciably displace a particle, an impulse from the trough of the wave tends to displace it in the opposite direction, and thus the particle remains at rest. As a consequence, there is a considerable *relative displacement* between the ether and the material particles. The actual reaction exerted by the stationary particles on the undulating ether is the same as if the ether were at rest, and the relative displacement were produced by a periodic motion of the particle parallel to the direction of wave displacement.

Let A (Fig. 257) be the position of a material particle in the path of transverse ether waves. The particle may be considered to possess no free period, and to be perfectly free to move in any direction under an impulse from the passing waves. It may consist of a number of molecules, and may, for instance, be a *very small* drop of water. If the wave displacements are in the plane of the paper and perpendicular to the horizontal line BC, the relative displacements between the particle and the ether will be in the line DE. To determine the effect produced, we may suppose that the ether is at rest, and the material particle vibrates, in a period equal to that of the waves, in the line DE.

This vibration will produce waves in the ether. If we consider a spherical wave surface, spreading out from A as centre, it is obvious that the displacement will be equal to zero at the points G and K, in a line with the direction of vibration of the particle; at these points, the only displacement which could be produced would be longitudinal; *i.e.* normal to the wave surface, or in the direction AG and AK, and the evidence at our disposal shows that the ether is incapable of transmitting longitudinal waves. The displacement will be a maximum at H and F, and here the displacement is parallel to the direction of vibration of the particle, or to the direction of the displacement in the primary waves. Draw a plane through FH perpendicular to the plane of the paper. At all points where this plane cuts the spherical surface FGHK, the displacement will be equal and parallel to that at F or H. The spherical wave surface, of which FGHK is the section, is due to the **scattering** of the primary wave by the small particle at A.

FIG. 257.—Scattering of Light by a Small Particle.

Let $a$ and $a'$ be the respective amplitudes of the primary and scattered waves. It is possible, by a simple method due to Lord Rayleigh, to determine the manner in which the ratio $a'/a$ varies with the length, $\lambda$, of the primary waves. The ratio $a'/a$ is, of course, a mere number, and cannot therefore depend on the primary units of length, mass, and time; in other words, $a'/a$ possesses no dimensions. Now the physical magnitudes involved in the problem are—

(1) The wave velocity, V.

(2) The volume, $v$, of the particle, which depends on the cube of the linear dimensions of the latter.

(3) The respective densities, $\rho'$ and $\rho$, of the particle and the ether.

(4) The distance, $r$, from the particle, to which the wave has travelled.

(5) The wave-length, $\lambda$, of the primary wave, which is equal to that of the scattered wave, since the periods of both are equal.

Now V is the only magnitude involving the unit of time, so that if V

occurred in the value of $a'/a$, the unit of time would be involved. The above ratio does not, therefore, depend on V. Since the unit of mass is not involved, $\rho'$ and $\rho$ can only occur in the form of a ratio $(\rho'/\rho)$, which is a mere number, and need not be considered. We are now left with $v$, $r$, and $\lambda$, on which quantities the ratio $a'/a$ must depend. The amplitude of a spherical wave varies inversely as its distance from the point of origin (p. 276); thus $a'/a \propto 1/r$. It is obvious that the amplitude of the wave, for a given displacement of the particle, will be directly proportional to $v$, the volume of the particle. Thus $a'/a \propto v/r$. But $v/r$ consists of a quantity involving the cube of a length, divided by a length; its dimensions therefore involve the square of a length. In order to get rid of the length dimensions we must divide by a length squared, and the only remaining length at our disposal is $\lambda$. Thus, finally—

$$a'/a \propto \frac{v}{r\lambda^2}$$

**In words, for a given amplitude of the primary wave, the amplitude of the scattered wave varies inversely as the square of the wave-length, and the intensity of the scattered light varies inversely as the fourth power of the wave-length.**

The student should be careful to realise the difference between the *reflection* and the *scattering* of light. A reflected wave is formed by the reinforcement of wavelets, generated at neighbouring points of a surface on which the primary wave is incident. The surface must therefore have dimensions at least comparable with the wave-length of light. When light is scattered, only a single wavelet is formed, the particle at which scattering occurs being small in comparison with a wave-length of light, and the ordinary laws of reflection are not obeyed.

Let us now consider a parallel pencil of white light, travelling horizontally from south to north, through air in which fine particles are suspended. To fix our ideas, the fine particles may be supposed to be due to smoke such as that from a cigarette; the light will be considered to be polarised in such a manner that its vibrations are vertical. The scattered light will consist mostly of blue and violet rays, since the intensity varies inversely as the fourth power of the wave-length. Consequently, on looking sideways (say in an east to west direction) at the pencil, the path of the light will be seen by means of the blue light scattered from the particles. Thus, the blue colour of thin

smoke, or weak milk and water, is explained. But no light will be propagated in a vertical direction from the particles, since the displacements in the primary wave are vertical, and therefore cannot give rise to horizontal displacements, which alone could be propagated vertically. Thus, **the path of the light will be invisible when looked at from above or below.**

Tyndall found that these phenomena are actually observed when polarised light passes through a glass tube containing fine particles in suspension. He allowed air to bubble through nitrite of butyl, and then to pass into an exhausted tube. A small amount of air, which had bubbled through hydrochloric acid, was then allowed to enter the tube till the pressure rose to a little more than half an inch of mercury. On allowing light from an arc lamp to traverse the tube, the path of the beam became gradually luminous, acquiring a beautiful azure tint, which in time merged into white. The action of the light provoked a chemical reaction between the butyl nitrite and the hydrochloric acid, and a cloud was formed, of which the constituent particles slowly increased in size ; when of suitable size, the violet and blue rays were scattered, giving the azure coloration to the path of the beam. When the tube was horizontal, and the light was polarised in a horizontal plane (p. 474) before entering it, the path of the beam was found to be invisible when viewed from above or below, but was brightly luminous, when viewed horizontally. The displacements in the incident polarised light must, then, have been vertical, since no light was scattered in a vertical direction ; and thus we see that light polarised in a horizontal plane comprises wave displacements executed only in a vertical direction, or **the direction of displacement is perpendicular to the plane of polarisation.** On examining the light scattered in a horizontal direction, it was found to be polarised in a horizontal plane, as we might anticipate from theory.

Professor Tyndall, to whom we are indebted for the above beautiful experiment, was unable to explain it on the wave theory ; the explanation given above was subsequently furnished by Lord Rayleigh.

**The Blue of the Sky.**—Newton considered that the blue of the sky might be produced by the interference of light reflected from small transparent particles suspended in the atmosphere.

It was subsequently proved that a blue coloration could only be produced if the light passed through thin transparent *sheets*, and even then the tint of the blue would be different from that of the sky. Brewster found that the light from the sky is polarised, and Tyndall suggested that the coloration is due to the scattering of light from *very small* water particles suspended in the upper atmosphere. There is now little doubt that the blue of the sky is derived from scattered light, but the particular substance which produces the scattering has not been determined. Lord Rayleigh considers that a good case could be made out for fine particles of common salt, or even for the oxygen in the atmosphere.

The blue tint of a distant mist, or of the smoke from a wood fire, must have attracted every one's attention. Some rivers have an intensely blue colour, due to finely divided particles of chalk held in suspension by the water. It must be remembered that when a wave is scattered at a particle, the energy of the scattered wave is derived from the primary wave, and after a sufficient amount of scattering the primary wave will be extinguished. Thus, a ray of sunlight, which has travelled a sufficient distance through the atmosphere, will be partially or wholly robbed of the blue and violet rays, and the remaining light will be of a yellow or red colour. This accounts for the colour of the clouds at sunset or sunrise, and the beautiful tints of snow-clad mountains at these periods.

**Wiener's Experiment.**—We have seen (p. 423) that when light is reflected normally at a perfectly reflecting surface, the waves approaching and receding from the surface combine to form stationary undulations. In certain circumstances similar results are obtained when light is reflected obliquely from a polished surface. Let AB (Fig. 258) be the section of a polished silver surface perpendicular to the plane of the paper. Let CD be the section of one of a number of parallel plane wave fronts, also perpendicular to the plane of the paper, incident at an angle of 45° on AB. The reflected wave front, CE, derived from CD, will also be inclined at 45° to AB ; and if CD represents a wave crest, and a phase change equal to $\pi$ occurs at reflection, CE will represent a wave trough. In Fig. 258, wave crests are represented by continuous lines, and wave troughs by broken lines. Each incident wave front, as it travels along AB, gives

rise to a reflected wave front differing from it in phase by $\pi$, and the space above the reflecting surface will be occupied by the two sets of waves as shown in Fig. 258. At the points marked by small circles, a crest of an incident wave is superposed on a trough of a reflected wave, or *vice versâ*. These points lie in straight lines parallel to AB, and correspond to planes parallel to the reflecting surface. The superposition of a crest on a trough will produce zero displacement, *if the displacements in the two waves are in the same straight line*. Now, if the displacements in the incident and reflected waves are perpendicular to the plane of incidence (*i.e.* perpendicular to the plane of the paper), this condition will be fulfilled. If, on the other hand, the displacements are *in* the plane of incidence, the incident wave displacements will be parallel to CD, and the reflected wave displacements will be parallel to CE, so that at a point where a crest and a trough are superposed, the resultant displacement will be due to two component displacements at right angles to each other, and interference will not occur. If, then, the displacements are perpendicular to the plane of incidence, a number of equidistant **nodal planes** will be formed parallel to the surface; between two consecutive nodal planes will be a region of maximum disturbance, where crests are superposed on crests, and troughs on troughs. If the displacements are in the plane of incidence, no interference will occur, and no nodal planes will be formed. The perpendicular distance between consecutive nodal planes is equal to $\lambda/\sqrt{2}$, and is therefore very small. An imaginary plane inclined at a very small angle to the reflecting surface will cut the nodal planes in parallel straight lines, and by diminishing the angle of inclination of the cutting plane, the distance between consecutive lines of intersection can be made as large as we please.

FIG. 258.—Illustrates Wiener's Experiment.

We have already seen that silver salts are not decomposed in nodal planes (p. 424). This circumstance was utilised by Wiener

to determine the conditions under which nodal planes are formed. He obtained a perfectly transparent film of silver chloride in collodion, on a sheet of glass. The thickness of the film was not more than a thirtieth part of a wave-length of light. Polarised light was reflected at an angle of 45° from a plane polished silver mirror, and the collodion film was placed in front of the mirror, being inclined to the latter at a little less than four minutes of arc. When the light was polarised in the plane of incidence, it was found that, after development, the photographic film was crossed by equidistant dark bands, separated by transparent spaces. The transparent spaces, in which the silver salts had been unacted upon, corresponded to the intersections of the nodal planes. When the light was polarised perpendicular to the plane of incidence, the film was uniformly blackened. Thus, nodal planes were formed only when the light was polarised in the plane of incidence; and since theory shows that nodal planes can be formed only when the displacements are perpendicular to the plane of incidence, it follows that **the displacements in polarised light are perpendicular to the plane of polarisation**, the result previously deduced by Lord Rayleigh from Tyndall's experiments.

Wiener also proved that a phase change amounting to $\pi$ occurs when light is reflected from a denser medium. A photographic film such as above described was placed in contact with the convex surface of a lens, and light was allowed to fall normally on the film through the glass plate on which it was supported. Since the film was very thin in comparison with a wave-length of light, a black spot surrounded the point of contact between film and lens. On developing the film, a circular clear space was found to surround the point of contact, thus showing that a node was formed at that point. This proves that a phase change amounting to $\pi$ occurs when the light, travelling through the film, is reflected at the optically denser glass (p. 283).

**Resolution of Displacements.**—It now becomes clear that the waves reflected at the polarising angle from a transparent surface are characterised by displacements perpendicular to the plane of incidence, or parallel to the surface of the glass. Waves in which the displacements are in the plane of incidence, and so, as it were, *cut into* the glass, are wholly transmitted when incident at the polarising angle. Any wave front will, of course, cut the plane of incidence in a straight line. If the direction of displacement in the wave front makes an angle,

$\theta$, with this line, and $a$ is the amplitude of the wave, then the latter may be resolved into a wave of amplitude $a \cos\theta$, vibrating in the plane of incidence, and another of amplitude $a \sin\theta$, vibrating perpendicular to the plane of incidence. The former wave will be entirely transmitted when incident at the polarising angle, and the latter will be partly reflected and partly transmitted. Thus, if we consider ordinary light to be characterised by vibrations performed indifferently in all directions in the wave front, it is plain that each vibration may be resolved in, and perpendicular to, the plane of incidence, and the components vibrating in the plane of incidence are entirely transmitted when the angle of incidence is equal to the polarising angle, so that the reflected light comprises only waves vibrating perpendicular to the plane of incidence.

**Double Refraction.**—In 1669, a Danish philosopher, Erasmus Bartholinus, discovered that a ray of light, when incident on a crystal of calcite, is not refracted according to the ordinary law, but forms *two* refracted rays, one of which does not of necessity lie in the same plane as the incident ray and the normal to the refracting surface.

Calcite (otherwise known as Iceland spar) is a transparent crystalline form of calcium carbonate, and was at one time found in great quantities in Iceland. It crystallises in many forms, each of which may be reduced, by cleavage, to a rhombohedron, bounded by six similar parallelograms with angles equal to 101° 55′ and 78° 5′ (Fig. 259). Two opposite solid angles, $a$ and $b$, are contained by three obtuse angles, while each of the remaining solid angles is contained by one obtuse and two acute angles. Let a line be drawn from either of the points $a$ or $b$, so as to be equally inclined to the three edges meeting there. This line, or any line drawn parallel to it, is termed the **axis of the crystal**. If the crystal is cleaved so that all of its edges are equal in length, the line joining the obtuse solid angles $a$ and $b$ will give the direction of the axis.

FIG. 259.—Crystal of Calcite.

On looking through a crystal of calcite placed over an illuminated pin-hole, two bright images are seen. If the eye is vertically above the crystal, and the latter is rotated, one image remains stationary, and is termed the **ordinary image.**

The second image is displaced from the first in a direction parallel to the shorter diagonal of the rhombic face through which it is observed ; this image rotates with the crystal, and is termed the **extraordinary image.**

EXPT. 78.—Lay a small crystal of calcite over an ink spot on a sheet of paper, and observe the two images formed, and the rotation of the extraordinary image with the crystal.

**Polarisation by Double Refraction.**—When a crystal of tourmaline is laid over one of calcite, and an illuminated pin-hole is viewed through the combination, it is found that for a certain position of the axis of the tourmaline, only the ordinary image formed by refraction through the calcite is seen. If the calcite crystal is kept stationary, and the tourmaline is rotated, both images come into view, the ordinary image at first being the brighter. When the tourmaline has been rotated through $45^{\circ}$, both images are equally bright ; with a further rotation the ordinary image gradually fades, and, when the tourmaline has been rotated through a right angle, only the extraordinary image is visible. But light which is transmitted through a tourmaline crystal in one position, and is intercepted when the tourmaline is rotated through a right angle, must be polarised. Consequently, the rays by which the ordinary and extraordinary images are seen must be polarised ; and since the extraordinary rays are intercepted by the tourmaline when the ordinary rays are most freely transmitted, and *vice versâ*, it follows that the ordinary and extraordinary rays are polarised in perpendicular planes. In other words, the wave displacements in the ordinary and extraordinary rays are executed at right angles to each other.

EXPT. 79.—View an illuminated pin-hole through a calcite crystal placed in front of the pile of plates made in Expt. 77, and notice that a rotation of the combination about the direction of vision makes no difference, but a rotation of the crystal *or* the pile of plates produces the changes described above.

The polarisation of light by transmission through calcite was discovered by Huyghens, soon after Bartholinus had published his discovery of double refraction. Huyghens was, however, unable to explain polarisation, since he supposed light to consist of waves in which the

displacements are longitudinal, or parallel to the direction of propagation. It was not till Young and Fresnel introduced the idea of waves in which the displacements are transverse that an explanation of polarisation was forthcoming.

**Refraction of Polarised Light by Calcite.**—The polariscope represented in Fig. 254 (p. 473) is furnished with a horizontal plate of glass, R, placed above the polarising mirror $A_1$. For the present we may suppose the mirror $A_2$ to be removed. If light is reflected at the polarising angle from $A_1$, so as to pass vertically upward through a pin-hole in a sheet of tinfoil laid on R, and then through a crystal of calcite, the refraction of polarised light through the calcite may be studied. The direction of the axis of the crystal has already been defined. Imagine a plane, perpendicular to the parallel refracting surfaces of the crystal, and passing through the axis or any parallel line. This plane is termed a **principal plane** of the crystal. The light reflected from $A_1$ is polarised in the plane of incidence, or in the plane containing the lines *ab*, *b*S. This plane is therefore the **plane of polarisation** of the light. When the crystal is placed so that its principal plane is parallel to the plane of polarisation, it is found that only the ordinary image is seen on looking in the direction S*b*. Consequently, **the ordinary ray is polarised in the principal plane.** If the crystal is rotated till its principal plane is perpendicular to the plane of polarisation, only the extraordinary image is seen. Consequently, **the extraordinary ray is polarised perpendicularly to the principal plane.** Since the direction of the displacement in polarised light is perpendicular to the plane of polarisation, it follows that light-waves, in which the displacements are parallel to the principal plane of the crystal, are transmitted along the extraordinary ray, and those in which the displacements are perpendicular to the principal plane are transmitted along the ordinary ray. To put this result in a slightly different form, **the vibrations in the wave front of the extraordinary ray are executed in a plane containing the transmitted ray and the axis of the crystal, while those in the wave front of the ordinary ray are executed at right angles to the plane containing the transmitted ray and the axis of the crystal.**

**Law of Malus.**—Let us now suppose that, in the wave front of the incident polarised ray, the direction of vibration makes an angle $\theta$ with the principal plane of the calcite. Let $a$ be the

amplitude of the vibration in the wave front of the incident ray; then the incident wave vibration is equivalent to a vibration of amplitude $a \cos \theta$, performed parallel to the principal plane of the calcite, combined with a vibration of amplitude $a \sin \theta$, performed at right angles to the principal plane of the calcite. The vibration of amplitude $a \cos \theta$ is transmitted along the extraordinary ray, and the intensity in this ray is proportional to $a^2 \cos^2 \theta$ (p. 276). The vibration of amplitude $a \sin \theta$ is transmitted along the ordinary ray, and the intensity in this ray is proportional to $a^2 \sin^2 \theta$. The sum of the intensities of the transmitted rays is therefore proportional to—

$$a^2(\cos^2 \theta + \sin^2 \theta) = a^2;$$

that is, **the sum of the intensities of the transmitted rays is equal to the intensity of the incident ray.**

Unpolarised light may be supposed to consist of waves in which the direction of displacement changes many times in a second. We can observe only the resultant effects produced by the waves in intervals of time during which many changes occur; in one of these intervals the direction of displacement in the incident unpolarised ray assumes all possible directions with respect to the principal plane of the calcite. Consequently, the average energies (or intensities) of the ordinary and extraordinary rays are equal, and are together equivalent to the average energy of the incident ray. Thus, if $A_o$ and $A_e$ are the effective amplitudes of the ordinary and extraordinary rays, and $a$ is the effective amplitude of the incident ray, we have—

$$A_o^2 = A_e^2 = a^2/2. \quad \therefore \; A_o = A_e = a/\sqrt{2}.$$

Let us now suppose that two similar crystals of calcite are laid face to face, and a pin-hole, illuminated by unpolarised light, is viewed through the combination. When the two crystals are arranged with their principal planes parallel, one ordinary and one extraordinary image is formed just as if a single crystal of the thickness of the two had been used. A small rotation of one of the crystals brings two extra images into view. If $a$ is the effective amplitude of the incident unpolarised light, the effective amplitudes of the ordinary and extraordinary rays, after transmission through the first crystal, are each equal to $a/\sqrt{2}$. Let $\theta$ be the inclination of the principal planes of the crystals. Then the extraordinary ray E transmitted by the first crystal gives rise to an extraordinary ray $E_e$, and an ordinary ray $E_o$, after transmission through the second crystal. The amplitude of $E_e$ is equal to $(a \cos \theta)/\sqrt{2}$, while

the amplitude of $E_o$ is equal to $(a \sin \theta)/\sqrt{2}$. The ordinary ount for this, mitted by the first crystal, gives rise to an ordinary ray ave surfaces extraordinary ray $O_e$, after transmission by the second c ation of the amplitude of $O_e$ is equal to $(a \sin \theta)/\sqrt{2}$, and the amplitu equal to $(a \cos \theta)/\sqrt{2}$. Thus, unless $\theta$ is equal to zero or so of $\pi/2$, we shall see four images, one pair being gener than the other pair. The intensity of either of the four c rays is proportional to the square of its amplitude. The fo renders the formation of these images clear :—

INTENSITIES OF INCID

| Incident ray. | Transmitted by first crystal. |
|---|---|
| | $O = \frac{1}{2}a^2$ . . |
| | $\frac{1}{2}a^2$ . . |



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any Direction.—We have, up to the present, at is transmitted through a natural crystal of may, however, be cut in the form of a parallel f which make any required angle with the axis Generally speaking, phenomena similar to the ted when light is transmitted normally through when the plate is cut perpendicular to the optic 1 that light is transmitted normally through it urrence of either polarisation or double refrac- agreement with the conclusions already arrived e circumstances *any* plane perpendicular to the ate will be parallel to the transmitted ray and and therefore vibrations in all of such planes can

fa the op be transı

**Huyghens's Theory of Double Refraction.**—Although Huyghens was unable to explain the polarisation of light produced by transmission through Iceland spar, he proposed a correct theory of double refraction. It is found that the ordinary ray obeys the ordinary laws of refraction : the incident and refracted

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with the normal to the refracting surface, lie in a ..., and the sines of the angles of incidence and re...r a constant ratio to each other. The refraction ...ary ray takes place, therefore, in the manner already ...p. 302); the refracted wave front is formed by the ...nforcement of spherical wavelets originating at the ...surface, as the incident wave front sweeps along it. ...ordinary ray does not obey the ordinary laws of ...; it does not, of necessity, lie in the same plane as the ...ay and the normal to the surface, and the sines of ...s of incidence and refraction bear no constant ratio ...each other. The direction of the extraordinary refracted ray depends, not only on the directions of the incident ray and the normal to the refracting surface, but also on the *direction of the axis of the crystal.* If the angle of refraction depends on the velocity with which waves are propagated within the refracting medium, it follows that the extraordinary wave velocity must depend on the direction of propagation relative to the optic axis. In other words, the properties of a doubly refracting medium, with respect to the propagation of light-waves, are different in different directions. Now, in a substance like glass, which has the same optical properties in all directions, a disturbance of the ether produces only a spherical wavelet. In a doubly refracting medium, the generation of spherical wavelets will account for the ordinary ray; to account for the extraordinary ray, Huyghens supposed that a second wavelet is also produced, which travels outwards with different velocities in different directions. He assumed the form of this wavelet to be the next in order of simplicity to the sphere, *i.e.* the spheroid or ellipsoid of revolution. This is the surface generated by the rotation of an ellipse about its major or minor axis, just as a sphere is generated by the rotation of a circle about a diameter. Since the properties of a crystal are symmetrical with respect to the optic axis, the axis of revolution of the spheroid is assumed to coincide with the optic axis of the crystal. When light is refracted at various angles through a crystal of Iceland spar, the ordinary and extraordinary rays make smaller and smaller angles with each other as their directions approximate to that of the optic axis; along the optic axis the two rays merge into one. It may therefore be inferred that the ordinary and extraordinary wave velocities

are equal in the direction of the optic axis. To account for this, it is assumed that the spherical and spheroidal wave surfaces touch each other at the extremities of the axis of rotation of the generating ellipse. We now have a complete picture of the two wave surfaces generated by a disturbance at any point in a doubly refracting medium. In the case of crystals like Iceland spar, the spheroid is oblate, *i.e.* the generating ellipse rotates about its shorter diameter. Consequently, the spherical sheet of the wave surface is entirely enclosed by the ellipsoidal sheet. Fig. 260 represents the sections of the two sheets by three mutually perpendicular planes, all of which pass through the common centre of the sphere and ellipsoid, while two intersect in the axis of the ellipsoid. A crystal possessing this form of wave surface is termed *negative*. In certain crystals, such as quartz, which are termed *positive*, the ellipsoidal sheet of the wave surface is entirely within the spherical sheet (Fig. 261). In this case the ellipsoid is prolate, being formed by the rotation of an ellipse about its major diameter, the latter being equal in length to the diameter of the spherical wave sheet.

FIG. 260.—Wave Surface of Negative Uniaxal Crystal.

We can now follow the process of double refraction in a crystal.

FIG. 261.—Wave Surface of Positive Uniaxal Crystal

Let LB (Fig. 262) represent the surface of a doubly refracting crystal, supposed to be perpendicular to the plane of the paper. Let AC be the trace of an incident plane wave, also perpendicular to the plane of the paper. At the instant when the incident wave front passes

A, two wavelets start from that point and spread out within the crystal. One wavelet has the form of a sphere, and enlarges equally in all directions; the other wavelet has the form of an ellipsoid, its axis of symmetry being parallel to the optic axis of the crystal. In Fig. 262, the axis of symmetry of the ellipsoid is represented by AF, so that the optic axis of the crystal is supposed to be parallel to the plane of the paper. This is merely done for convenience of representation; in the general case the optic axis may possess any direction, and the axis of symmetry of the ellipsoid must be drawn, in imagination, parallel to that direction.



FIG. 262.—Illustrates Double Refraction by a Negative Uniaxal Crystal.

At the instant when the incident wave reaches B, having travelled over the distance CB after passing A, let the spherical and ellipsoidal wavelets from A occupy the respective positions DEFG and HKFL. From B draw the line BE so as to touch the circle DEFG. A plane through BE perpendicular to the paper will touch all the spherical wavelets generated between A and B (p. 303), and this plane represents the **ordinary wave front**. Further, AE represents the **ordinary refracted ray** corresponding to the incident ray MA. Through B draw a plane perpendicular to the plane of the paper, so as to touch the ellipsoid HKFL; then this plane touches all the ellipsoidal wavelets generated between A and B, and constitutes the **extraordinary wave front**. When the axis of the ellipsoid is either in, or perpendicular to, the plane of the paper, the extraordinary wave front touches the ellipsoid at a point K in the plane of the paper; but in the general case the point of contact of the wave front with the ellipsoid may be either above or below the plane of the paper. The ellipsoidal wavelets generated in the immediate neighbourhood of the point A intersect each other in the point where the plane wave front BK touches the ellipsoid HKFL. Thus, wavelets generated near A reinforce each other at K, and AK is the **extraordinary ray** corresponding to the incident ray MA. When the incident ray lies in the same plane as the normal to the surface and the optic axis of the crystal, the extraordinary ray lies in the

plane of incidence; but in the general case the extraordinary ray is inclined to the plane of incidence. It must be noticed that the extraordinary wave front BK is not, in general, perpendicular to the extraordinary ray AK.

In negative crystals, the extraordinary ray is less refracted than the ordinary ray, since the extraordinary wave velocity is greater than the ordinary wave velocity. On drawing a diagram similar to Fig. 262, with reference to a positive crystal, it is easily seen that in this case the extraordinary ray is more refracted than the ordinary ray.

In Fig. 262, the direction of vibration in the ordinary wave front BE will be perpendicular to the plane of the paper (p. 487). In the extraordinary wave front, the direction of vibration will be parallel to KB, in the plane of the paper.

**Verification of Huyghens's Construction.**—The construction described above obviously gives a qualitative explanation of the phenomena of double refraction; experiments made by Huyghens, Wollaston, Stokes, Mascart, and Glazebrook prove that the construction is also quantitatively exact. The general methods of experimenting are as follows:—

1. CONSTRUCTION FOR ORDINARY RAY.—A number of parallel slices are cut from Iceland spar, in various directions with respect to the optic axis. These slices are cemented together, and from the combination a prism is cut with its refracting edge perpendicular to the planes of junction (Fig. 263). When this prism is mounted on the table of a spectrometer, each slice of the prism produces a separate extraordinary spectrum, but all of the slices combine to produce a single ordinary spectrum of a character similar to that obtained by the use of a glass prism. Thus, the direction of the ordinary ray is independent of the direction of the optic axes, and the ordinary wave surface is similar to that in glass, or has the form of a sphere.

FIG. 263.—Compound Prism.

2. CONSTRUCTION FOR ORDINARY AND EXTRAORDINARY RAYS, WHEN THE OPTIC AXIS IS PARALLEL TO THE REFRACTING SURFACE. —When the optic axis is parallel to the refracting surface, and perpendicular to the plane of incidence, the latter cuts the wave surface in two concentric circles (compare Figs. 260 and 261, p. 491). In this case both the ordinary and extraordinary rays lie in the plane of incidence, and the

angle of refraction of the extraordinary ray bears a constant ratio to the angle of incidence. Thus, if a prism is cut from a doubly refracting crystal, so that the refracting edge is parallel to the optic axis, two spectra will be formed. Either of these spectra may be examined by itself, by observing the transmitted light through a tourmaline crystal with its axis in a suitable direction. Thus, the refractive indices for the ordinary and extraordinary rays can be separately determined. The refractive index for the ordinary ray will be the same for all directions of transmission through the crystal. The refractive index for the extraordinary ray will only apply to the case where the plane of incidence is perpendicular to the optic axis.

Let $V_o$ be the velocity of transmission in the ordinary wave front, while $V_e$ is the velocity of transmission in the extraordinary wave front, in a direction at right angles to the optic axis. Then, if V is the velocity of light *in vacuo*, the **ordinary refractive index**, $\mu_o$, of the crystal is equal to $V/V_o$. Similarly, the **extraordinary refractive index**, $\mu_e$, is equal to $V/V_e$. It is obvious that $V_o$ and $V_e$ are proportional to the equatorial semi-diameters of the spherical and ellipsoidal sheets of the wave surface. In the case of a negative crystal, $V_e$ is greater than $V_o$, so that $\mu_e$ is less than $\mu_o$. The opposite is the case with respect to a positive crystal. Having determined the value of $\mu_e$ and $\mu_o$, we are in a position to construct the complete wave surface. For, if we describe a sphere of unit radius to represent the spherical sheet of the wave surface, then the polar semi-diameter of the ellipsoid will be equal to unity, and the equatorial semi-diameter will be equal to $V_e/V_o = V/V_o \div V/V_e = \mu_o/\mu_e$.

3. Construction for the Ordinary and Extraordinary Rays, when the Optic Axis is in the Plane of Incidence.—Having determined the major and minor semi-diameters of the ellipsoidal wave sheet, and the radius of the spherical sheet, we can draw the section of the wave surface by the plane of incidence for any position of the optic axis in that plane (Fig. 262). We can then determine, graphically or by calculation, the angles of refraction for the ordinary and extraordinary rays, corresponding to any given angle of incidence. Light from a pinhole may be refracted through a parallel plate of the crystal, arranged so that the optic axis lies in the plane of incidence: by observing the position of the extraordinary image of the pin-hole, we can determine the angle of refraction into, or out of, the crystal, and so verify the results obtained from theory.

An elaborate series of experiments conducted by Mr. Glazebrook, has shown that Huyghens's construction gives results which agree with those observed to within 1 in 30,000.

The following table gives the principal refractive indices of Iceland spar and quartz, for the chief Fraunhofer lines in the spectrum : —

| Fraunhofer lines. | Iceland spar. | | Quartz. | |
|---|---|---|---|---|
| | $\mu_e$ | $\mu_o$ | $\mu_e$ | $\mu_o$ |
| B . . . . . . | 1·48391 | 1·65308 | 1·54990 | 1·54090 |
| C . . . . . . | 1·48455 | 1·65452 | 1·55085 | 1·54181 |
| D . . . . . . | 1·48635 | 1·65850 | 1·55328 | 1·54418 |
| E . . . . . . | 1·48868 | 1·66360 | 1·55631 | 1·54711 |
| F . . . . . . | 1·49075 | 1·66802 | 1·55894 | 1·54965 |
| G . . . . . . | 1·49453 | 1·67617 | 1·56365 | 1·55425 |
| H . . . . . . | 1·49780 | 1·68330 | 1·56772 | 1·55817 |

**Nicol's Prism.**—ABCD (Fig. 264) is the isometric projection of a long crystal of Iceland spar, of which B and D are the obtuse solid angles, each contained by three obtuse plane angles. An imaginary plane drawn through the blunt edges BC and AD contains the optic axis of the crystal (p. 485); this plane may be termed the **principal section** of the crystal. A ray incident on the face AB, in a plane parallel to the principal section, gives rise to an ordinary and extraordinary ray, both of which lie in the plane of incidence. These rays travel with different velocities, that of the extraordinary ray being the greater.

The refractive index of Canada balsam is equal to 1·55, which is a value intermediate between the ordinary and extraordinary refractive indices for Iceland spar. Consequently, the velocity of wave transmission in Canada balsam is greater than the ordinary, and less than the extraordinary, velocity of wave transmission in the spar. If the ordinary ray in the spar is incident on a layer of Canada balsam, it will

FIG. 264.—Illustrates the Method of constructing a Nicol's Prism.

be totally refl . . the angle of incidence is greater than a certain critical v. . . . on the other hand, the extraordinary ray cannot be totally re. ected under similar conditions, since its velocity in the balsam is less than in the spar. Thus, by allowing the ord. and extraordinary rays, travelling through Iceland spar, to . . a layer of Canada balsam at a suitable angle, the ordin. . can be reflected to one side, and so got rid of, while . extraordinary ray is transmitted. This is the principle of co. . tion of Nicol's prism.

The natural end fac . AB . DC (Fig. 264) of a crystal of Iceland spar are inclined at al . 71 . the blunt edges AD and BC. In con- . ctin . a Nicol's prism, the crystal is first cut . . u to the principal section, so as to f. new end faces AE and CF inclined at al . 68 . the edges AF and EC. The prism is t . divi. d into two parts by a cut EGF, per- pe. lar . the principal section, and also to the . faces AE and CF. The cut surfaces are grou . plane, polished, and cemented together by a . of Canada balsam. The prism is then . plete . xcept for mounting in a brass tube t . . ect the spar from injury. Fig. 265 represe . the section of a Nicol's prism by a plane p . through the opposite blunt edges EC and . The film of balsam is represented by EF. . ray, HK, incident in the plane of the paper . end face AE, is divided into an ordinary . KO, and an extraordinary ray, KM, withi . the prism. The ordinary ray is totally refle. . at O from the film of balsam, while the . . rdinary ray is transmitted through the . , and leaves the face FC parallel to the . on HK of the incident ray. Thus, we obtai . ray of pure polarised light, of considerable in . . Since the vibrations in the extraord. . ray are executed in the principal plane (p. 487), it follows that . brations in the ray MN will be parallel to the plane of the pape . . 266 represents the end view of a Nicol's prism. The direct . vibration in the transmitted light is parallel to the shorter . al of the end face, or in the direction of the double arrow.

FIG. 265.—Section of Nicol's Prism.

A Nicol's prism (often termed, for brevity, a Nicol) can be used either as a polariser or as an analyser. When two Nicols are placed end to end, with their principal sections parallel, the light transmitted through one is also transmitted through the other. If one of the Nicols is rotated, the transmitted light gradually becomes feebler; the intensity of the transmitted light is proportional to the square of the cosine of the angle between the principal sections of the two Nicols (p. 488). When the principal sections are perpendicular to each other, no light is transmitted; the extraordinary ray from the first Nicol forms an ordinary ray in the second, and so is totally reflected from the balsam. When the Nicols are arranged with their principal sections perpendicular to each other, they are said to be crossed.

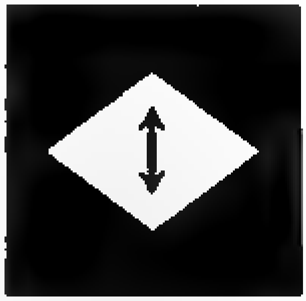

FIG. 266. End view of Nicol's Prism, showing Direction of Displacement in Transmitted Rays.

**The Double Image Prism.**—In some investigations it is of advantage to split a single ray of light into two divergent rays. This result can be accomplished by the use of a double image prism. Fig. 267 represents a section of Wollaston's prism. It consists of two right-angled prisms of quartz, or calcite, cemented together so as to form a prism of rectangular section. The prism ABD is cut so that the face AB is parallel, while the refracting edge B is perpendicular, to the optic axis. The prism BCD is cut so that the face CD and the refracting edge D are both parallel to the optic axis. Thus, as indicated by the shading, in the prism ABD the optic axis is in the plane of the paper, while in the prism BCD the optic axis is perpendicular to the plane of the paper.

FIG. 267.—Double Image Prism.

A ray incident normally on the face AB forms ordinary and extraordinary rays within ABD, which travel along the same path with unequal velocities. The vibrations in the ordinary ray are perpendicular to the plane of the paper. After passing the face BD, this ray is transmitted as an extraordinary ray in the prism BCD, since in BCD vibrations perpendicular to the plane of the paper will be parallel to the principal plane. If $\mu_o$ and $\mu_e$ are the ordinary and extraordinary refractive indices, the effective refractive index for the above refraction at the face BD is equal to $\mu_e/\mu_o$. The

extraordinary ray in ABD comprises only vibrations in the plane of the paper, and the ray is transmitted as an ordinary ray after passing the face BD. The effective refractive index for this refraction is equal to $\mu_o/\mu_e$. In the case of quartz, $\mu_e > \mu_o$, so that on passing the face BD the extraordinary ray in ABD is refracted toward the base AD of the prism ABD, while the ordinary ray in ABD is refracted away from the base AD. On emerging from the face CD, the two rays continue to diverge.

A double image prism may be used to measure the dimensions of an optical image, after the manner described with regard to the ophthalmometer (p. 162).

**Tourmaline, a Doubly Refracting Crystal.**—Only a single polarised ray is transmitted through a thick crystal of tourmaline; but if the crystal is thin, two rays of unequal intensities are transmitted, and these rays are found to be polarised in perpendicular planes. The directions of the rays are found to follow Huyghens's construction, so that we reach the conclusion that tourmaline is a doubly-refracting crystal, in which the ordinary ray is absorbed if the crystal is thicker than 1 or 2 mms., while the extraordinary ray is transmitted without much loss of intensity. Calcite acts like tourmaline with respect to very long waves. The transmissive power of calcite for infra-red rays of wave-lengths between 1 and 5·5 microns, has been studied by Merritt; he finds that beyond 3·2 microns, the ordinary ray is entirely absorbed, while the extraordinary ray is transmitted. Thus, double refraction is seen to be closely associated with absorption, and the latter property has been found to depend on the free periods of the material particles set in motion by light-waves (p. 376). If we suppose that the vibrating particles of a doubly refracting crystal are arranged regularly, so that their periods of vibration are different for displacements which are respectively parallel and perpendicular to the optic axis, then it follows that waves will be transmitted with different velocities according to the direction of displacement in the wave front (p. 386). This gives an explanation of double refraction, and the polarisation of the transmitted rays. If the period of the incident waves agrees with one of the free periods of the material particles, when vibrating in a particular direction, then the ray corresponding to this direction of vibration will be absorbed after traversing a small thickness of the crystal.

**Biaxal Crystals.**—In calcite, quartz, and similar crystals,

there is a single direction, termed the optic axis, in which all waves are transmitted with one uniform velocity. In any other direction there are two distinct velocities of wave transmission, and the resulting rays are polarised perpendicularly to each other. Brewster and Biot discovered another class of crystals, in which there are *two* distinct directions of single wave velocity; these directions are termed the **optic axes**, and such crystals are said to be **biaxal**. To these crystals Huyghens's construction does not apply. The refraction of light by biaxal crystals will be studied in detail in the next chapter.

**Elliptic and Circular Polarisation.**—A plate of uniaxal crystal, cut parallel to the axis, produces no separation in a ray incident on it normally. The extraordinary ray is transmitted normally with a velocity $V_e$, equal to $V/\mu_e$, where V is the velocity of light *in vacuo*, and $\mu_e$ is the extraordinary refractive index of the crystal; the vibrations in this ray are parallel to the axis of the crystal. Similarly, the ordinary ray is transmitted normally with a velocity $V_o$, equal to $V/\mu_o$, where $\mu_o$ is the ordinary refractive index of the crystal; in this ray the vibrations are perpendicular to the axis of the crystal. If the incident ray is polarised, and its vibrations have an amplitude $a$, and make an angle $\theta$ with the principal plane (*i.e.* the plane containing the normal to the surface and the optic axis of the crystal), then, by the law of Malus, the amplitude of the extraordinary ray is equal to $a \cos \theta$, while that of the ordinary ray is equal to $a \sin \theta$. Unless $\theta = 45°$, the amplitudes of the ordinary and extraordinary rays will be unequal. Thus, in general, waves of unequal amplitudes are transmitted, with unequal velocities, along the common path of the two rays. On entering the crystal the phases of the ordinary and extraordinary wave vibrations are equal; but, owing to the unequal wave velocities, an increasing difference of phase is introduced during transmission. Each particle in the common path of the rays simultaneously executes two harmonic motions at right angles to each other; the periods of these harmonic motions are equal, but in general the amplitudes and phases differ, so that the actual path of a particle in general is an ellipse, including the circle and straight line as particular instances (p. 244).

To fix our ideas, let us suppose that the vibrations in the incident polarised ray make an angle of 45° with the principal plane of the crystalline plate. The amplitudes of the two rays are now equal, their

common value being $a/\sqrt{2}$. The plate may be supposed to be of quartz, so that the ordinary is greater than the extraordinary wave velocity. If $\lambda$ is the wave-length of the incident light, then, since the ordinary wave travels a distance $V_o$ in the plate while the incident light travels a distance V *in vacuo*, the wave-length in the ordinary ray is equal to $V_o\lambda/V$, or $\lambda/\mu_o$. Similarly, the wave-length in the extraordinary ray is equal to $\lambda/\mu_e$. Let us suppose that, at a distance $\delta_1$ within the plate, the extraordinary and ordinary wave displacements differ in phase by $\pi/4$. Remembering that a phase difference amounting to $\pi$ corresponds to a retardation of half a wave-length, we see that the extraordinary wave front has fallen behind the ordinary wave front by one-eighth of the extraordinary wave-length. Thus, if there are $n$ ordinary waves between the surface of the crystal and the point in question, there will be $(n + \frac{1}{8})$ extraordinary waves in the same space. Consequently—

$$\delta_1 = n\lambda/\mu_o = \left(n + \frac{1}{8}\right)\lambda/\mu_e.$$

$$\therefore\ n\lambda = \mu_o\delta_1 = \mu_e\delta_1 - \frac{\lambda}{8}.$$

$$\therefore\ \frac{\lambda}{8} = (\mu_e - \mu_o)\delta_1, \quad \text{and } \delta_1 = \frac{\lambda}{8(\mu_e - \mu_o)}.$$

At the point in question, let us suppose that the ordinary and extraordinary wave vibrations are respectively performed along the axes of $x$ and $y$ (Fig. 131, p. 245). Since the amplitudes are equal, the tracing points move round the same circle; but at the instant when the ordinary tracing point passes, in the positive direction, through the axis of $x$, the extraordinary tracing point has still to move through an angular distance equal to $\pi/4$, before reaching the axis of $y$. Applying the method explained with respect to Fig. 131, it is easily seen that the resultant of the two vibrations is an ellipse, described in a direction opposite to that in which the hands of a clock revolve (Fig. 268, II). Thus, at a distance $\delta_1$ within the plate, each particle describes an elliptic orbit. If the thickness of the plate is equal to $\delta_1$, so that the two rays leave it after traversing this thickness, then, since no further phase change occurs after the light emerges, each particle of the ether in the path of the transmitted ray describes an elliptic orbit. The transmitted light is then said to be **elliptically polarised.**

If the plate is of sufficient thickness, we may find a point at a distance $\delta_2$ within it, such that the phases of the ordinary and extraordinary wave displacements differ by $\pi/2$. The value of $\delta_2$ is given by—

$$\delta_2 = \lambda/4(\mu_e - \mu_o),$$

or the optical difference between the paths of the ordinary and extraordinary rays [*i.e.* $(\mu_e - \mu_o)\delta_2$] is equal to $\lambda/4$. At the point in question the particle describes a circular orbit (Fig. 268, III). If the plate has a thickness of $\delta_2$, each particle of the ether in the path of the emergent light describes a circular orbit. In this case the emergent

FIG. 268.—Composition of two mutually Rectangular Vibrations differing in Phase by various amounts.

light is said to be **circularly polarised**, and the crystalline plate is termed a **quarter wave plate**.

The student should now find no difficulty in verifying the following results :—

At a distance $\delta_3$ within the plate, given by—

$$\delta_3 = 3\lambda/8(\mu_e - \mu_o),$$

the phase difference amounts to $3\pi/4$, and the orbit of a particle is an ellipse, its major axis being at right angles to the major axis of the elliptic orbit at a distance $\delta_1$ within the crystal (Fig. 268, IV).

At a distance $\delta_4$ within the plate, given by—

$$\delta_4 = \lambda/2(\mu_e - \mu_o),$$

the phase difference amounts to $\pi$, and the resultant vibration is in a straight line perpendicular to the original direction of vibration (Fig. 268, V). Still farther within the plate, we find points at which the phase difference amounts to $5\pi/4$, $3\pi/2$, $7\pi/4$, and $2\pi$. The orbits at these points are represented in Fig. 268, VI, VII, VIII,

and I. Notice that, for phase differences between $\pi$ and $2\pi$, the direction of revolution along the elliptic orbits is opposite to that for phase differences between o and $\pi$.

**Detection of Circularly or Elliptically Polarised Light.**—A circular vibration is equivalent to two equal linear vibrations, differing in phase by $\pi/2$, executed at right angles to each other. When a ray of circularly polarised light is analysed by a Nicol, it is resolved into two plane-polarised rays of equal amplitudes; the ray in which the vibrations are parallel to the principal section of the Nicol is transmitted, while the other ray is totally reflected from the balsam. A rotation of the Nicol produces no alteration in the intensity of the transmitted ray, since the amplitudes of the resolved rays are always equal. In this respect circularly polarised light resembles unpolarised light. To distinguish between the two, let the light be transmitted through a quarter wave plate (p. 501). Circularly polarised light will be decomposed into two rectilinear vibrations at right angles to each other, initially differing in phase by $\pi/2$; the quarter wave plate introduces a further phase change of $\pi/2$, so that on emergence the component vibrations differ in phase by $\pi$ or o, and in either case a single rectilinear vibration is the result (Fig. 268, I and V). This rectilinear vibration will be refused transmission by a Nicol, when the principal section of the latter is perpendicular to the direction of vibration. On the other hand, unpolarised light, after transmission through a quarter wave plate, will not be plane-polarised, and when analysed by a Nicol, will not be extinguished as the Nicol is rotated. Thus, we can distinguish between unpolarised and circularly polarised light.

An elliptic vibration may be resolved into two unequal rectilinear vibrations, in any two directions at right angles to each other. Accordingly, when elliptically polarised light is analysed by a Nicol, the intensity of the transmitted light is greatest when the principal section of the Nicol is parallel to the major diameter of the ellipse, and least when the principal section is parallel to the minor diameter of the ellipse. Thus, as the analysing Nicol is rotated, the intensity of the transmitted light alternately decreases and increases. In this respect elliptically polarised light resembles a mixture of unpolarised and plane-polarised

light. To distinguish between the two, let the light be transmitted through a quarter wave plate. When an elliptic vibration is resolved into rectilinear vibrations, parallel to its major and minor diameters, these vibrations differ in phase by $\pi/2$ (p. 246). If the quarter wave plate is arranged so that its axis is parallel to either the major or minor diameter of the elliptic vibration, such a resolution will occur, and a further phase change of $\pi/2$ will be introduced as the light traverses the plate, so that on emergence the component vibrations of the light differ in phase by $\pi$ or 0. The resultant vibrations will then be rectilinear, and the emergent light can be extinguished by a Nicol. Partially polarised light will not be rendered plane-polarised by transmission through a quarter wave plate. Thus, we have a means of distinguishing between partially polarised light and elliptically polarised light.

It should be noticed that, owing to the unequal variations of $\mu_o$ and $\mu_e$ with the wave-length (Table, p. 495), a plate of quartz can only serve as a quarter wave plate for a particular wave-length of light.

**Rotation of the Plane of Polarisation.**—As already explained, no light is transmitted through crossed Nicols. If a plate of calcite, cut perpendicular to the axis, is placed between crossed Nicols, a parallel pencil of light is still refused transmission; the polarised light transmitted by the first Nicol is transmitted without modification by the calcite, and is extinguished by the second Nicol. When, however, a plate of *quartz*, cut perpendicular to the axis, is placed between crossed Nicols, it is found that light is transmitted through the combination. If the light is monochromatic, the ray leaving the quartz can be extinguished by rotating the analysing Nicol through a definite angle. Thus, the light transmitted by the quartz is plane-polarised. In passing through the quartz the plane of polarisation has been rotated through a definite angle. Using quartz plates, cut in a similar manner from different crystals, it is found that the analysing Nicol must sometimes be rotated in one direction, and sometimes in the opposite, in order to extinguish the transmitted light. Thus, there are two kinds of quartz, distinguished from each other by the directions in which they rotate the plane of polarisation. Look along the ray in the direction opposite to that

**in which the light travels; then, if the plane of polarisation is rotated in the direction in which the hands of a clock revolve, the crystal is said to be right-handed. A left-handed crystal rotates the plane of polarisation in the opposite direction.**

Biot investigated the rotation of the plane of polarisation by quartz. He deduced the following laws :—

1. The rotation is proportional to the thickness of the crystal traversed.

2. The rotation produced by transmission through two plates is the algebraic sum of the rotations due to the separate plates. Thus, a plate of right-handed quartz just neutralises the rotation produced by a plate of left-handed quartz of equal thickness. Further, if a polarised ray, after being transmitted through a quartz plate, is reflected back normally by a polished silver mirror, the rotation produced by the first transmission through the plate is just neutralised by the second transmission through it in the opposite direction. In both cases the rotation is, let us say, right-handed; but in the first we must look along the incident ray in the direction opposite to that in which the light travels, while in the second we must look along the returning ray.

3. The rotation varies with the wave-length of the transmitted light; to a first approximation it is inversely proportional to the square of the wave-length. Thus, the rotation is much greater for blue or violet, than for red, rays.

**Fresnel's Explanation.**—We have seen (p. 243) that a rectilinear vibration is equivalent to two circular vibrations executed in opposite directions. Fresnel assumed that a ray of plane-polarised light, incident normally on a plate of quartz cut perpendicular to the axis, is decomposed into two circularly polarised rays, which are transmitted along the axis with unequal velocities.

Let OA (Fig. 269) be the amplitude of the incident wave vibrations. At the face of the quartz plate the rectilinear vibrations, performed along AB, are resolved into two circular vibrations performed in opposite directions around the circle CDEGC, where OC = OA/2. The tracing points pass each other at C and E, the right-handed circular vibration being executed in the direction CDEGC, while the left-handed circular vibration is executed in the direction CGEDC. At any instant, the phase of either vibration is measured by the angle subtended at O by the circular arc described by the tracing point since its passage through C. At the surface of the quartz the phases of the two opposite circular

vibrations are equal. At a distance within the quartz, the phases of the two circular vibrations differ, if the right-handed and left-handed waves are transmitted with different velocities. For let $V_r$ and $V_l$ be the velocities with which the right- and left-handed waves are propagated, while T is the period of the incident wave vibrations. In every case a complete revolution of a tracing point is executed in a time T. The lengths of the right- and left-handed waves are respectively equal to $V_rT$ and $V_lT$. At any instant the phase of the right-handed vibration, at a distance $V_rT$ within the plate, will be $2\pi$ radians behind the right-handed vibration at the surface of the plate. Consequently, at a distance $x$ within the plate, the phase of the right-handed vibration will be $2\pi x/V_rT$ radians behind the right-handed vibration at the surface. In other words, when the right-handed tracing point at the surface is passing through C, the right-handed tracing point, at a distance $x$ within the crystal, has still to traverse an arc GC, which subtends an angle $2\pi x/V_rT$ at O, before reaching C. At the same instant, the left-handed tracing point at $x$ must traverse an arc FC, which subtends an angle $2\pi x/V_lT$ at O, before reaching C. If $V_r$ is less than $V_l$, GC is greater than FC. Since the tracing points describe their circular paths with equal velocities, the left-handed tracing point passes through C when the right-handed tracing point has still an arc HC, equal to (GC − FC), to traverse. The two tracing points pass each other at K, where KC = HC/2. Then OK is the direction of the resultant rectilinear vibration at a distance $x$ within the plate. The angle through which the plane of polarisation has been rotated is equal to KOC. Since the arc HC subtends an angle $\frac{2\pi x}{T}\left(\frac{1}{V_r} - \frac{1}{V_l}\right)$ at O, the arc KC subtends an angle $\frac{\pi x}{T}\left(\frac{1}{V_r} - \frac{1}{V_l}\right)$ at O, and this is the angle through which the plane of polarisation has been rotated by transmission through a thickness, $x$, of quartz. The rotation of the plane of polarisation is left-handed when $V_l > V_r$, and right-handed when $V_r > V_l$.

FIG. 269.—Illustrates Fresnel's Explanation of the Rotation of the Phase of Polarisation.

Fresnel proved the above explanation to be substantially

accurate in the following manner. A number of quartz prisms were cut so that, when cemented together in series, they formed a rectangular parallelepiped (Fig. 270), the axes of the quartz in all of the prisms being in the same direction, perpendicular to the end faces AC and EF. The prisms ABC and BDE were right-handed, while CBD and DEF were left-handed. A plane-polarised ray, incident normally on the face AC, is decomposed, according to Fresnel's theory, into two circularly polarised rays which travel through the first prism with unequal velocities. The right-handed ray travels faster in the first prism (which is right-handed) than in the second (which is left-handed); consequently it is refracted toward the base of the prism CBD. Conversely, the left-handed ray is refracted away from the base of CBD. This separation of the rays will be further increased by refraction through the third and fourth prisms, and two circularly polarised rays, diverging from each other, should leave the combination. Fresnel realised these results experimentally.

FIG. 270.—Arrangement of Prisms to test Fresnel's Theory.

The late Sir George Airy proved that when a plane-polarised ray is transmitted through quartz, in a direction inclined to the axis, two elliptically polarised rays are transmitted with unequal velocities. In a direction perpendicular to the axis, the elliptic vibrations degenerate into two mutually perpendicular rectilinear vibrations.

**Rotations produced by Liquids and Vapours.**—Many liquids rotate the plane of polarisation of light in a manner similar to that described above. The rotatory power of liquids is generally smaller than that of solids. A plate of quartz 1 mm. thick rotates the plane through $21^\circ{\cdot}67$; a layer of turpentine of the same thickness rotates the plane through $0^\circ{\cdot}296$. Essence of lavender and cane-sugar solutions produce right-handed rotations; essence of turpentine and grape-sugar solutions produce left-handed rotations.

Liquids possessing rotatory power do not lose this property

when diluted with inactive liquids, and even in the state of vapour the rotatory power may be retained. On the other hand, quartz loses its rotatory power on being fused, or on being dissolved in caustic potash solution. The neutral anhydrous tartrate of rubidium has its rotatory power reversed by solution. In all cases the rotation produced by a substance varies inversely as the square of the wave-length of the transmitted light.

**Laurent's Saccharimeter.**—The angle through which the plane of polarisation is rotated, when light is transmitted through a known thickness of an active substance in solution, is proportional to the mass of the active substance per c.c. of solution. Consequently, the amount of a substance, such as cane sugar, in 1 c.c. of a solution, may be determined from an observation of the rotation of the plane of polarisation.

For accurate work it is not sufficient to polarise light by a Nicol, transmit it through a known length of the solution, and analyse it by a second Nicol; for the latter, when adjusted to intercept the light transmitted through the solution, can be rotated through a small angle in either direction without allowing any appreciable amount of light to pass. Laurent overcame this difficulty very successfully in his half-shade saccharimeter. Let AQB (Fig. 271) be a semicircular plate of quartz, cut so that the optic axis is in the plane of the paper parallel to AB. Plane-polarised light transmitted normally through the plate will be divided into two sets of waves travelling along the same path, but polarised in perpendicular planes. Let OP be the direction of the vibrations in the incident light. In the quartz the extraordinary ray vibrations are parallel, while the ordinary ray vibrations are perpendicular, to OA. Further, the ordinary and extraordinary waves travel through the quartz with unequal velocities. If the thickness of the quartz is such that, on emergence, the phases of the ordinary and extraordinary waves differ by $\pi$, then it can be proved, by a method similar to that used on p. 245, that the resultant vibrations are rectilinear, and are executed along OQ, where $\angle QOA = \angle POA$. For a phase difference of $\pi$ to be pro-

FIG. 271.—The Half-shade.

duced, the optical difference in the paths of the ordinary and extraordinary rays must be equal to half a wave-length of the incident light (p. 501); the plate AQB is then termed a **half wave plate.** Let APB be a semicircular sheet of glass of such thickness that it reflects and absorbs as much light as the quartz; the vibrations in the rays transmitted by the glass will be performed parallel to OP. Thus, the rays transmitted through the semicircle AQB are characterised by vibrations executed in the direction OQ, while the rays transmitted through the semicircle APB are characterised by vibrations executed in the direction OP. On viewing the circle APBQ through a Nicol, its halves AQB and APB will generally be unequally bright. On rotating the Nicol so that its principal section is perpendicular to AB, only the component vibrations perpendicular to AB will be transmitted, and, since OP and OQ are equally inclined to OA, the transmitted rays from APB and AQB will be of equal amplitude, and the two semicircles will appear equally bright. A slight rotation of the analysing Nicol, in either direction, produces a marked alteration in the brightnesses of the two semicircles. The combination of glass and quartz semicircles is termed a **half-shade.**

Fig. 272 represents Laurent's Saccharimeter; the parts are represented diagrammatically below. Light from a slit, *a*, is rendered parallel by a lens, *e*, and is polarised by a Nicol, *d*; it then illuminates the half-shade *f*. By means of the small Galilean telescope *ih*, which is focussed on the half-shade, the latter can be viewed through the analysing Nicol, *g*. The telescope *ih* and the Nicol *g* are carried by a tube, K, which is rotated by a milled head, G, its position being determined by means of a vernier and scale viewed through a lens, L. Having adjusted the analyser so that the half-shade appears uniformly bright, a tube, *pp*, filled with sugar solution, is placed between the half-shade and the analyser. The light from either portion of the half-shade has its plane of polarisation rotated through the same angle; on rotating the analyser a position can be found where the half-shade again appears uniformly bright; the angle through which the analyser has been rotated is equal to the angle through which the plane of polarisation is rotated by the solution in the tube *pp*. Owing to the circumstance that the half wave plate in the half-shade must be cut to a thickness dependent on the wave-length of the light to be used, it is usual to employ a Bunsen flame coloured with salt for a source of light; a plate of bichromate of potash at B cuts off all light except that corresponding to the D lines. The

line of intersection of the halves of the half-shade can be rotated through a small angle by the arm J, so as to vary the angle AOP (Fig. 271).

FIG. 272.—Laurent's Saccharimeter.

## QUESTIONS ON CHAPTER XVIII

1. What do you understand by plane-polarised light? Describe some form of apparatus for determining the plane of polarisation of a beam of polarised light.

2. Rays of light polarised in, and perpendicular to, the plane of incidence respectively are, in turn, reflected at different angles from glass. Describe the phenomena observed, and apply them to the explanation of polarisation by reflection.

3. Explain carefully how you would arrange a number of sheets of glass to act as a polariser and analyser, and describe any single experiment which might be performed with the aid of the apparatus described, to illustrate the properties of polarised light.

4. Describe a method of polarising light by reflection; and state how you would (1) prove that the light is polarised, and (2) determine whether it is completely polarised.

5. Define the plane of polarisation of a parallel beam of polarised light, and discuss the question as to whether the direction of vibration lies in, or at right angles to, the plane of polarisation.

6. The waves of light are said to be transverse. What is the evidence for this? Röntgen's rays do not appear to satisfy the ordinary tests for transversality. Is it thereby proved that they are longitudinal? Discuss this question.

7. Write a short essay on the colour of the sky.

8. How may Huyghens's construction for the form of the wave surface in a uniaxal crystal be verified by experiment?

9. A ray of light falls upon the surface of a uniaxal crystal, the plane of incidence being a principal plane. Give the geometrical construction for determining the paths of the rays within the crystal, and state whether your diagram is drawn for the case of a positive or negative crystal.

10. Give a geometrical construction for the direction of the refracted rays in a crystal of Iceland spar, when the optic axis is the intersection of the plane of incidence and the refracting surface. Explain the figure you draw.

11. Describe the construction of a Nicol's prism, and explain how it produces plane-polarised light. How may a beam of circularly polarised light be produced, and how may it be distinguished from a beam of ordinary light?

12. Explain what is meant by the ordinary and extraordinary rays in a crystal, defining any technical terms you employ, and describing simple experiments by which your description of the two kinds of rays is justified.

13. A horizontal beam of sunlight enters a dark room through a small hole, and passing through a properly placed crystal of tourmaline becomes polarised. If the tourmaline were made to rotate rapidly about an axis coincident with the ray, state and explain the appearance you would see if you looked at it through a Nicol's prism.

14. Enumerate the different kinds of polarised light, and explain how they may be distinguished from one another, and from common light.

15. The plane of polarisation of light traversing a plate of quartz cut perpendicular to the axis is rotated, and the rotation is inversely proportional to the square of the wave-length. Describe experiments to verify these statements.

16. Describe the phenomenon of rotatory polarisation, such as that exhibited by sugar solutions ; and also describe some form of instrument for measuring the strength of sugar solutions by means of this property.

17. Discuss the method of producing, and testing for, circularly polarised light.

18. From the table on p. 495, calculate the thickness of a quarter wave plate of quartz, for the D rays ($\lambda = 589 \times 10^{-7}$ cm.).

## PRACTICAL

1. Find the index of refraction of the given opaque plate by measuring the polarising angle.

2. Determine the rotation of the plane of polarisation of soda light per mm. of a plate of quartz traversed by light parallel to its axis.

3. Determine the sign and magnitude of the rotation of the plane of polarisation produced by 1 cm. of the given solution.

4. To determine the proportion of sugar present in a syrup by the rotation of the plane of polarisation.

## CHAPTER XIX

### THEORIES OF REFLECTION AND REFRACTION

IN the present chapter a short account will be given of some attempts which have been made to explain the phenomena attending the reflection and refraction of light, in terms of the properties of an elastic solid. Fresnel's theories, which will chiefly concern us, are not dynamically sound, since certain of his assumptions are at variance with exact mechanical principles, and there are, moreover, inconsistencies amongst the assumptions themselves. But the results obtained are in very close agreement with the experimental evidence at our disposal; in far better agreement, indeed, than those of many later and more consistent theories.

**Isotropic and Æolotropic Media.**—A substance is said to be **homogeneous**, when all parts of it are exactly alike; otherwise, it is said to be **heterogeneous**. Thus, glass is homogeneous, while granite, which is a mixture of small portions of quartz and mica in a matrix of felspar, is heterogeneous. A homogeneous substance may possess different properties in different directions. Thus, a piece of rolled metal will have different tensile strengths along and across the direction in which it was rolled. Substances which possess the same properties in all directions are said to be **isotropic**; those possessing different properties in different directions are said to be **æolotropic** or **anisotropic**. A substance may be isotropic with respect to certain physical agencies, and æolotropic with respect to others. Thus, many crystals are æolotropic with regard to light, producing polarisation, which differs in different directions; but all attempts to detect any difference in the gravitative attraction

of a crystal in different directions have proved futile. A substance naturally isotropic may be rendered æolotropic by certain physical agencies ; thus, glass, when submitted to mechanical strain, behaves toward light in some respects like a crystal.

## ISOTROPIC MEDIA

**General Conditions.**—In order to account for the transmission of light through space, we assume the existence of an all-pervading, imponderable medium termed the *luminiferous ether;* through this medium waves are transmitted, and when these waves possess periods lying between certain limits, they constitute light. The phenomena of polarisation force us to conclude that the direction of vibration of the ether particles is parallel to the wave-front, and, in isotropic media, perpendicular to the direction of the ray. Hence, has arisen the theory that the ether possesses properties similar to those of an elastic solid. The spaces between the molecules of material substances are supposed to be occupied by the ether, and ethereal vibrations tend to move the material molecules, the reactions of the latter affecting the properties of the ether, either as regards its effective density or its effective elasticity.

**Fresnel's Theory of the Reflection of Light at the Plane Surface of a Transparent Isotropic Medium.**—We have seen that light reflected at a certain angle from the plane surface of a transparent medium is polarised in the plane of incidence, while the refracted light is partially polarised in a perpendicular plane ; and, that there is satisfactory experimental evidence that the vibrations of the polarised light are perpendicular to the plane of polarisation. Thus, the vibrations of the reflected polarised beam are perpendicular to the plane of incidence, while those of the refracted polarised beam are in the plane of incidence. In Fresnel's time experimental evidence on this latter point was altogether wanting ; with rare insight, however, he made the correct assumption in this respect.

His second assumption, which is merely an instance of the universal law of the conservation of energy, was that in a given time the energy carried up to the surface of the medium by an incident pencil of light, is equal to the energy carried away from it by the corresponding reflected and refracted pencils.

In Fig. 275, let AB, CD, and CE, be the incident, reflected, and refracted wave-fronts of parallel pencils of light. Then, if $v_1$ is the velocity of light in the upper medium, the energy contained by a length $v_1$ of the incident pencil will reach the reflecting surface, and the energy contained in an equal length of the reflected pencil will travel away from it, in one second. If $v_2$ is the velocity of light in the lower medium, the energy carried away from the surface in a second through this medium will be that corresponding to a length $v_2$ of the refracted pencil. Let $a$, $b$, $c$, be the respective amplitudes in the incident, reflected, and refracted pencils. Then, if $\rho_1$, $\rho_2$, are the respective densities of the upper and lower media, the energy per unit volume of the incident, reflected, and refracted pencils will be respectively proportional to $\rho_1 a^2$, $\rho_1 b^2$, and $\rho_2 c^2$ (p. 276).

FIG. 273.—Incident, Reflected, and Refracted Pencils.

Thus, since the breadth, perpendicular to the plane of the paper, of all three pencils will be equal, we have—

Energy of length $v_1$ of Incident pencil $\propto \rho_1 a^2 v_1 \,.\, AB.$
,, ,, ,, Reflected ,, $\propto \rho_1 b^2 v_1 \,.\, CD.$
,, ,, $v_2$ of Refracted ,, $\propto \rho_2 c^2 v_2 \,.\, CE.$

Let $i$ and $r$ be the angles of incidence and refraction. Then, $\angle BAC = \angle DCA = i$, and $\angle ACE = r$.

Also, $AB = CD = AC \cos i$, and $CE = AC \cos r$.

Then, according to the law of conservation of energy, we have—

$$\rho_1 a^2 v_1 \,.\, AC \cos i = \rho_1 b^2 v_1 \,.\, AC \cos i + \rho_2 c^2 v_2 \,.\, AC \cos i.$$

$$\therefore \rho_1 v_1 (a^2 - b^2) \cos i = \rho_2 v_2 c^2 \cos r. \quad . \quad . \quad . \quad . \quad (1)$$

Fresnel's next assumption was that the elasticity of the ether is the same in all media, so that the velocity with which light will be transmitted through them will vary inversely as the square roots of their optical densities (p. 273). Thus—

$$\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}; \quad \therefore \rho_1 v_1^2 = \rho_2 v_2^2.$$

$$\therefore \frac{\rho_1 v_1}{\rho_2 v_2} = \frac{v_2}{v_1}$$

If the index of refraction from the upper to the lower medium is equal to $\mu$, then $v_1/v_2 = \mu$, and we have—

$$\frac{\rho_1 v_1}{\rho_2 v_2} = \frac{v_2}{v_1} = \frac{1}{\mu} = \frac{\sin r}{\sin i}.$$

Substituting in (1), we obtain—

$$(a^2 - b^2)\cos i \frac{\sin r}{\sin i} = c^2 \cos r.$$

$$\therefore\ a^2 - b^2 = c^2 \tan i \cot r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have thus obtained one equation between the quantities $a$, $b$, and $c$. In order to determine $b$ and $c$ in terms of $a$, we need yet another equation.

In order to obtain this second equation, Fresnel reasoned as follows:—At any point in a medium traversed by waves, the displacement of an ether particle will be produced by the joint action of the various waves passing through that point. In the upper medium (Fig. 273), there will be two trains of waves, corresponding to the incident and reflected light pencils; while, in the lower medium, there will be only one train of waves, corresponding to the refracted pencil. At two points, indefinitely near to, and on opposite sides of, the surface of separation of the media, the component displacements parallel to the surface must be equal, since otherwise there would be slipping between the ether particles on opposite sides of the surface. Let us assume that incident waves, vibrating in the plane of incidence, give rise to reflected and refracted waves, also vibrating in the plane of incidence; while incident waves, vibrating perpendicularly to the plane of incidence, give rise only to waves vibrating perpendicularly to the plane of incidence. Further, let us assume that no phase change, other than can be denoted by a change of sign (*i.e.* other than a phase change of $\pi$), occurs at reflection or refraction. Then, if the sum of the component amplitudes of the incident and reflected waves, resolved parallel to the surface of separation, is equal to the amplitude of the refracted waves, resolved in the same direction, the displacements on opposite sides of the surface will be equal, and no slipping will occur. The application of this principle will vary, according as the incident waves vibrate in, or perpendicularly to, the plane of incidence.

**Light Polarised in the Plane of Incidence.**—In this case the vibrations in the incident, reflected, and refracted wave fronts will all be perpendicular to the plane of incidence, *i.e.* perpendicular to the plane of the paper in Fig. 273. Just above the refracting surface, the resultant displacement is equal to the

L L 2

sum of the incident and reflected wave displacements, *i.e.* to $(a + b)$. Just below the refracting surface the displacement is equal to $c$. Thus, for there to be no slipping at the surface—

$$a + b = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Dividing (2) by (3), we obtain—

$$a - b = c \tan i \cot r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Adding (3) and (4) we obtain—

$$2a = c(1 + \tan i \cot r) = c\frac{\cos i \sin r + \sin i \cos r}{\cos i \sin r} = c\frac{\sin(i + r)}{\cos i \sin r}.$$

$$\therefore c = \frac{2a \cos i \sin r}{\sin (i + r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Subtracting (4) from (3), we obtain—

$$2b = c(1 - \tan i \cot r) = c\frac{\cos i \sin r - \sin i \cos r}{\cos i \sin r} = - c\frac{\sin (i - r)}{\cos i \sin r}.$$

$$\therefore b = - a\frac{\sin (i - r)}{\sin (i + r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the refracted ray is inclined to the normal at a smaller angle than the incident ray (*i.e.*, $i > r$), then the second medium is optically denser than the first, and $\sin (i - r)$ is positive. Also, as $(i + r)$ cannot exceed 180°, $\sin (i + r)$ must be positive. Thus, if at any instant the incident wave displacement at the surface is in one direction, the reflected wave displacement will be in an opposite direction, since the signs of $b$ and $a$ are opposite; or there is a change of phase of $\pi$ on reflection at a denser medium. When reflection occurs at a rarer medium, $i < r$, and $\sin (i - r)$ is negative, so that the signs of $a$ and $b$ are similar, and there is no change of phase.

When the angle of incidence is small, we may substitute the circular measures of $(i - r)$ and $(i + r)$ for the sines of these quantities, and $\cos i = 1$. Thus, remembering that $i = \mu r$ in this case, we have, for normal incidence,—

$$\left.\begin{aligned} b &= - a\frac{i - r}{i + r} = - a\frac{\mu - 1}{\mu + 1}. \\ c &= 2a \cdot \frac{r}{i + r} = 2a \cdot \frac{1}{\mu + 1}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The intensity of a pencil will be proportional to the rate at which energy travels normally across an area of 1 sq. cm.; *i.e.*, to the product of the velocity, the density, and the square of the amplitude (p. 276).

Thus, the intensities of the incident, reflected, and refracted pencils at normal incidence, will be respectively proportional to $a^2$, $a^2\left(\frac{\mu - 1}{\mu + 1}\right)^2$, and $\mu a^2\frac{4}{(\mu + 1)^2}$, since (p. 514) $\frac{v_2\rho_2 c^2}{v_1\rho_1} = \frac{v_1}{v_2}c^2 = \mu c^2$. This result has been verified photometrically by Arago, and for thermal radiations by by Prevostaye and Desains.

**Light Polarised Perpendicularly to the Plane of Incidence.**—In this case the vibration in the wave-front will be in the plane of incidence, or along the lines AB, DC, EC (Fig. 273). The positive direction for $a$ is from A to B (Fig. 273), while the positive directions of $b$ and $c$ are respectively from D to C and from E to C.

Since $\angle$ BAC = $\angle$ DCA = $i$, and $\angle$ ECA = $r$, the components of the incident and reflected wave displacements resolved parallel to AC will be equal to $a \cos i$ and $b \cos i$, while the component of the refracted wave displacement resolved parallel to AC will be equal to $c \cos r$. Hence, to determine $c$ and $b$ in terms of $a$, we have—

$$(a + b) \cos i = c \cos r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

$$a^2 - b^2 = c^2 \tan i \cot r. \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Dividing (9) by (8), we obtain—

$$\frac{a - b}{\cos i} = c\frac{\tan i}{\sin r}.$$

$$\therefore a - b = c\frac{\sin i}{\sin r}.$$

Also, from (8)—

$$a + b = c\frac{\cos r}{\cos i}.$$

$$\therefore 2a = c\left(\frac{\sin i}{\sin r} + \frac{\cos r}{\cos i}\right) = c\frac{\sin i \cos i + \sin r \cos r}{\cos i \sin r}$$

$$= \frac{c}{2}\frac{\sin 2i + \sin 2r}{\cos i \sin r} = c\frac{\sin (i + r) \cos (i - r)}{\cos i \sin r},$$

according to the general formula—

$$\sin A + \sin B = 2 \sin\frac{A + B}{2} \cos\frac{A - B}{2}.$$

$$\therefore c = 2a\frac{\cos i \sin r}{\sin (i + r) \cos (i - r)}. \quad . \quad . \quad . \quad . \qquad (10)$$

Also,—

$$2b = c\left(\frac{\cos r}{\cos i} - \frac{\sin i}{\sin r}\right) = c\,\frac{\sin r\cos r - \sin i\cos i}{\cos i\sin r} = \frac{c}{2}\,\frac{\sin 2r - \sin 2i}{\cos i\sin r}$$

$$= -\,c\,\frac{\cos(i+r)\sin(i-r)}{\cos i\sin r}.$$

According to the general formula—

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Therefore, from (10)—

$$b = -\frac{c}{2}\cdot\frac{\cos(i+r)\sin(i-r)}{\cos i\sin r} = -\,a\cdot\frac{\tan(i-r)}{\tan(i+r)}. \qquad (11)$$

When $(i + r)$ is less than 90°, $\tan(i + r)$ will be positive. In these circumstances, the signs of $b$ and $a$ will be opposite when the second medium is denser than the first, so that $i > r$. This denotes a change of phase amounting to $\pi$ on reflection from a denser medium.

When the incident light is nearly normal to the surface, the sines and tangents of $(i + r)$ and $(i - r)$ may be put equal to the circular measures of the corresponding angles, while the cosines of $(i - r)$ and $i$ will be equal to unity. Thus—

$$c = 2a\cdot\frac{r}{i+r} = 2a\,\frac{1}{\mu+1},$$

and

$$b = -\,a\,\frac{i-r}{i+r} = -a\,\frac{\mu-1}{\mu+1},$$

results similar to those obtained for light polarised in the plane of incidence. When light is incident normally (or practically normally) on a surface, all vibrations will be parallel to the surface, since they are executed in the wave-front which is parallel (or approximately parallel) to the surface. In this case the same result is obtained for the reflected and refracted rays, in whatever plane the incident light may be polarised.

When $(i + r) = \frac{\pi}{2}$, $\tan(i + r) = \infty$, and

$$b = -\,a\,\frac{\tan(i-r)}{\tan(i+r)} = -\,a\,\frac{\tan(i-r)}{\infty} = 0.$$

In this case $\mu = \dfrac{\sin i}{\sin r} = \dfrac{\sin i}{\sin\left(\frac{\pi}{2} - i\right)} = \tan i.$

**Thus, when the vibrations of the incident light are in the plane of incidence, and the angle of incidence, $i$, is such that $\tan i = \mu$, or $(i + r) = \pi/2$, the light will be wholly refracted, and no reflected ray will be produced.** (Compare p. 475.)

The amplitude of the refracted ray will then be equal to—

$$c = 2a\frac{\cos i \sin r}{\sin(i + r)\cos(i - r)} = 2a\frac{\sin^2 r}{1 \times \cos\left[i - \left(\frac{\pi}{2} - i\right)\right]} = 2a\frac{\sin^2 r}{\sin 2i}$$

$$= 2a \cdot \frac{\sin^2 r}{2\sin i\cos i} = a\frac{\sin^2 r}{\sin i \sin r} = a\frac{\sin r}{\sin i} = \frac{a}{\mu}.$$

The intensity (p. 516) of the refracted ray is equal to—

$$c^2 = \mu(a/\mu)^2 = a^2/\mu.$$

Equal amounts of energy pass per second through the planes AB and CE (Fig. 273), since $EC/AB = \cos r/\cos i = \sin i/\cos i = \mu$.

As the value of $(i + r)$ passes through 90°, the sign of $\tan(i + r)$ changes from + to −. Thus, when $(i + r)$ is just below 90°, $a$ and $b$ will have opposite signs when the second medium is the denser; when $(i + r)$ just exceeds 90°, $a$ and $b$ will have similar signs in the same circumstances. Thus, **with light vibrating in the plane of incidence, a change of phase amounting to $\pi$ occurs in the reflected light as the angle of incidence passes through the angle of polarisation.**

Expt. 80.—Touch a water surface with a glass rod that has been dipped in turpentine ($\mu = 1{\cdot}5$ nearly). The turpentine spreads over the water, and exhibits the colours of thin films (p. 401). Observe the colours through a Nicol arranged so as to transmit only the waves vibrating in the plane of incidence, and gradually increase the angle of incidence; the colours disappear when the angle is equal to 56·5° ($\tan 56{\cdot}5° = 1{\cdot}5$ nearly), and the complementary colours appear when this angle is exceeded. At an angle of 56·5° the rays incident on the free turpentine surface are polarised; to be polarised at the turpentine-water interface, the rays would have to be incident on that interface at an angle $\tan^{-1}(1{\cdot}33/1{\cdot}5) = 41°$, and therefore their angle of incidence on the free turpentine surface would be $\sin^{-1}(1{\cdot}5 \sin 41°) = 80°$.

**Light Polarised in any Plane.**—When the vibrations of the incident light are executed neither in, nor perpendicular to, the plane of incidence, we may resolve them in, and perpendicular to, the plane of incidence, and treat these component vibrations according to the above methods. It is clear that, as the angle of incidence approaches the angle of polarisation, the component

of the reflected wave vibrating in the plane of incidence will be diminished, and will finally vanish as that angle is reached. Thus, the effect of reflection of polarised light is to bring the vibrations of the reflected light more and more nearly perpendicular to the plane of incidence as the polarising angle is approached, and therefore the plane of polarisation of the reflected ray is rotated toward the plane of incidence as the angle of incidence approaches the angle of polarisation.

**Reflection and Refraction of Unpolarised Light.**—It is probable that the vibrations constituting unpolarised light are not performed continuously in any particular plane ; at a particular point in space, the vibrations of the passing waves may be performed in a definite plane for a certain time, and then an abrupt change in the direction of vibration occur. As the orange light of the spectrum consists of waves of which about 500 billion ($500 \times 10^{12}$) pass a particular point during a second, it is clear that many of these abrupt changes may occur during a second, while a large number of similar vibrations may still be performed consecutively.

The effect of these abrupt changes in the direction of vibration of the incident light will be inappreciable by the eye, if they occur at sufficiently small intervals of time.

When unpolarised light is reflected from the surface of a transparent medium, the component vibrations perpendicular to the plane of incidence will always be reflected to a greater extent than those in the plane of incidence. From the investigation already performed, we see that the ratio of the amplitudes of the reflected waves, consisting respectively of vibrations in, and perpendicular to, the plane of polarisation, will be equal to—

$$\frac{\tan(i-r)}{\tan(i+r)} \div \frac{\sin(i-r)}{\sin(i+r)} = \frac{\cos(i+r)}{\cos(i-r)};$$

and the ratio of the corresponding intensities will be equal to—

$$\frac{\cos^2(i+r)}{\cos^2(i-r)}.$$

While $(i+r)$ is less than 90°, $\cos(i+r)$ will always be less than $\cos(i-r)$. Thus, there will be an excess of light polarised in the plane of incidence (vibrating perpendicular to

that plane) in the beam reflected from the surface of a transparent medium.

For an angle of incidence $i$, given by the equation $\tan i = \mu$, the light vibrating in the plane of incidence will be wholly transmitted, and the reflected light will consist wholly of vibrations perpendicular to the plane of incidence. This is in accordance with Brewster's law (p. 475), and accounts for polarisation by reflection. The two polarised beams will be of equal intensities, since the sum of the component vibrations resolved parallel to the plane of incidence will on an average be equal to the sum of the components resolved perpendicular to that plane.

**Total Internal Reflection.**—If light is refracted into a rarer medium, then we may write $\mu_1 \sin i = \sin r$, where $\mu_1$ is the relative refractive index of the first (denser), with respect to the second (rarer), medium. Now, for vibrations perpendicular to the plane of incidence—

$$b = a\frac{\sin(r-i)}{\sin(i+r)} = a\frac{\cos i \sin r - \sin i \cos r}{\cos i \sin r + \sin i \cos r}$$

$$= a.\frac{\mu \cos i \sin i - \sin i\sqrt{1-\mu_1^2 \sin^2 i}}{\mu \cos i \sin i + \sin i\sqrt{1-\mu_1^2 \sin^2 i}} \quad . \quad . \quad (12)$$

For the value of $b$ to be real, we must have the quantities under the radical signs equal to zero or a positive number. Hence, the greatest angle of incidence for which the laws previously deduced will hold without modification, will be given by the equation—

$$1 - \mu_1^2 \sin^2 i = 0, \quad \text{or} \quad \sin i = \frac{1}{\mu_1}.$$

Substituting in (12), we get—

$$b = a\frac{\mu_1 \cos i \sin i - 0}{\mu_1 \cos i \sin i + 0} = a.$$

Thus, **the light is totally reflected at the angle of incidence $i$, given by the equation,** $\sin i = 1/\mu_1$.

It would at first sight appear necessary that the amplitude,

$c$, of the refracted ray should in these circumstances become equal to zero. But

$$c = 2a\frac{\cos i \sin r}{\sin (i + r)} = 2a\frac{\cos i \sin r}{\cos i \sin r + \sin i \cos r}$$

$$= 2a\frac{\mu \cos i \sin i}{\mu \cos i \sin i + \sin i \sqrt{1 - \mu_1^2 \sin^2 i}}.$$

Hence, when $\mu_1 \sin i = 1$, $c = 2a$.

In interpreting this result, it must be recollected that experiment proves that, in the circumstances considered, a disturbance really does travel into the rarer medium, but dies out within a wave-length or so from its surface (p. 412). Hence, we see that the above value of $c$ must be taken as referring to the amplitude of this superficial disturbance. No appreciable amount of energy passes into the rarer medium, owing to the small distance to which the disturbance penetrates. When $i = 90°$, $\cos i = 0$, and the value of $c$, the amplitude of the refracted superficial disturbance becomes equal to zero. Thus $c$ diminishes as the angle of incidence is increased from its critical value to 90°. When $\mu \sin i > 1$, the values of $b$ and $c$ become complex, *i.e.* they take the form $A + \sqrt{-1} . B$.

To understand the meaning of this, it must be remembered that we have assumed that no change of phase occurs in the reflected or refracted rays, other than that which corresponds to a reversal of sign (*i.e.* to an acceleration or retardation by $\pi$). This assumption we have found to lead to consistent results *except when light is incident on a rarer medium, the angle of incidence being greater than the critical angle.* If, in this case, we assume the occurrence of phase changes in the reflected and refracted lights, varying continuously with the angle of incidence, we can here also obtain consistent results.

The general nature of this assumption can be understood by referring to Fig. 274.

Let a line, OA′, rotate about O in a time equal to the period of vibration of the incident light-waves. Then, if $OA = a$, the displacement of an ether particle at the surface of separation of the two media, due to the incident light, will at any instant be given by Oa, the horizontal projection of OA′. A similar construction may be used to

determine the displacement of an ether particle due to the reflected or refracted light, the period of vibration being in all three cases the same. Now, assuming that there is no change of phase in the reflected or refracted light, the equation, $a + b = c$, means that the line of length $c$, which must rotate about O, in order to give the displacement in the refracted pencil, must be equal to OA + AB, where OA = $a$, and AB = $b$. But, if a change of phase occurs, let the angle COB indicate the difference in phase between the refracted and incident waves. Then the line OC must be the *resultant* of OA and AC, where AC is the amplitude of the reflected wave, so that the angle CAB must denote the phase change in the reflected wave. In the case of total reflection, the numerical lengths of OA and AC must be equal (since $a = b$), and therefore, as the amplitude of the superficial refracted disturbance decreases, with an increase in the angle of incidence, so as to reach the value zero when $i = 90°$, the phase change of the reflected light must increase.

FIG. 274.—Phase Changes in the Reflected and Refracted Waves.

Reasoning based on the principles explained above shows that in the case of light polarised in a plane perpendicular to the plane of incidence, as in the case of light polarised in the plane of incidence, the phase change due to internal reflection increases from 0 to $\pi$, as the angle of incidence increases from its critical value to $\pi/2$. But for an angle of incidence between these limits, the phase change due to internal reflection of light polarised in the plane of incidence will differ from that of light polarised in a plane perpendicular to the plane of incidence. Fresnel calculated that, in the case of glass, internal reflection at an angle of incidence equal to 55° would produce a difference in these phase changes, amounting to $\pi/4$.

**Fresnel's Rhomb.**—To test his conclusions, Fresnel constructed a rhomb of glass (Fig. 275) such that a ray of light,

AB, could enter normally at one end, and then, after being twice internally reflected at equal angles of incidence of 55°, should emerge normally from the opposite end. If the incident light is polarised, its vibrations making an angle of 45° with the plane of incidence, then the component vibrations, resolved perpendicular and parallel to the plane of incidence, will be equal. A phase difference of $\pi/4$ should be introduced between the two sets of vibrations at each reflection, so that the emergent light should consist of two equal trains of waves, vibrating at right angles to each other, and differing in phase by $\pi/2$. Thus, every particle of the ether in the path of the emergent light should move in a circular orbit, in the plane of the wave-front (p. 501). In other words, the emergent light should be circularly polarised.

FIG. 275.—Fresnel's Rhomb.

This was found to be the case. Moreover, as was anticipated, if the entering light is circularly polarised, a further change of phase amounting to $\pi/2$ is introduced between the component vibrations, so that the total phase difference amounts to $\pi$, and the emergent light is plane-polarised, in a direction making an angle of 45° with the plane of incidence.

If elliptically polarised light is passed through a Fresnel's rhomb, the axes of the elliptic vibrations being respectively in, and perpendicular to, the plane of incidence, a further difference of phase amounting to $\pi/2$ is introduced between the component vibrations, which already differ by $\pi/2$ (p. 246), so that plane-polarised light is produced in this case also.

**Theory of MacCullagh.**—Fresnel assumed, as the condition to be satisfied at the surface of separation of two media, that the displacements, *parallel to the surface*, of ether particles on opposite sides of it, should be equal, so as to avoid slipping at the surface. But it would appear equally necessary that the displacements, *perpendicular to the surface*, should be equal, and in the same direction; otherwise there would be a separation between ether particles on opposite sides of the

surface. MacCullagh worked out the results of the inclusion of this latter condition.

Let us first consider the application to **vibrations in the plane of incidence.**

Let the positive direction of the amplitude $a$ of the incident light be from A to B (Fig. 273). To determine the positive direction of the amplitude $b$ of the reflected light, notice that, at normal incidence, AB will coincide with DC; thus, the positive direction of $b$ is from D to C. Similarly, the positive direction of $c$ is from E to C. Now, the component of $a$, resolved perpendicularly to the surface, will be equal to $a \sin \mathrm{BAC} = a \sin i$. This will correspond to a displacement *upwards* from the surface. The component of $b$, resolved perpendicularly to the surface, will be equal to $b \sin \mathrm{DCA} = b \sin i$, directed *downwards* toward the surface. Hence, the displacement of an ether particle near the surface, in the upper medium, will be proportional to $(a - b) \sin i$, directed *upwards* from the surface.

Similarly, $c \sin r$ will be proportional to the displacement of an ether particle near the surface, but in the lower medium; the direction of the displacement in this case also will be upwards. Hence, in order to avoid separation at the surface, we have—

$$(a - b) \sin i = c \sin r \quad . \quad . \quad . \text{(no separation)} \quad . \qquad (13)$$

Also—

$$(a + b) \cos i = c \cos r \quad . \quad . \quad . \quad . \quad . \text{(no slipping)} \quad . \qquad (14)$$

Multiplying these equations together, we obtain—

$$(a^2 - b^2) \sin i \cos i = c^2 \sin r \cos r \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Substituting $v_1/v_2 = \sin i/\sin r$ in the energy equation (1), p. 514, we obtain—

$$\rho_1(a^2 - b^2) \sin i \cos i = \rho_2 c^2 \sin r \cos r \quad . \quad . \quad . \qquad (16)$$

Equations (15) and (16) can only be rendered consistent by writing $\rho_1 = \rho_2$. *Thus, in order that there should be neither slipping nor separation at the surface, the densities of the two media must be equal,* the differences in the velocity of light being due to differences in the elasticity of different media.

Solving (13) and (14) for ($b$) and ($c$) in terms of $a$, we obtain—

$$a (\sin i \cos i + \cos i \sin i) = c (\sin r \cos i + \cos r \sin i) = c \sin (i + r).$$

$$\therefore \; c = \frac{2a \sin i \cos i}{\sin (i + r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

$$b(\cos i \sin i + \sin i \cos i) = c(\cos r \sin i - \sin r \cos i) = -c \sin(i - r).$$

$$\therefore\ b = -c\frac{\sin(i-r)}{2\sin i \cos i} = -a\frac{\sin(i-r)}{\sin(i+r)}. \quad . \quad . \quad (18)$$

Equations (17) and (18), according to MacCullagh's theory, give the amplitudes of the refracted and reflected waves, *when the vibrations are in the plane of incidence;* they are identical with Fresnel's equations, (5) and (6), p. 516, *referring to vibrations perpendicular to the plane of incidence,* except that the value of $c$ is increased in the ratio $\sin i/\sin r = \mu$.

For **vibrations perpendicular to the plane of incidence**, we have—

$$a + b = c,$$

since the vibrations are parallel to the surface, and therefore no separation can take place. Combining this with MacCullagh's energy equation (15), we obtain—

$$(a - b)\sin i \cos i = c \sin r \cos r.$$

$$\therefore\ 2a = c\left(1 + \frac{\sin r \cos r}{\sin i \cos i}\right) = \frac{c}{2}\left(\frac{\sin 2i + \sin 2r}{\sin i \cos i}\right)$$

$$= c\,\frac{\sin(i+r)\sin(i-r)}{\sin i \cos i}.$$

$$\therefore\ c = 2a\frac{\sin i \cos i}{\sin(i+r)\sin(i-r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Also—

$$2b = c\left(1 - \frac{\sin r \cos r}{\sin i \cos i}\right) = \frac{c}{2}\,\frac{\sin 2i - \sin 2r}{\sin i \cos i} = -c\,\frac{\cos(i+r)\sin(i-r)}{\sin i \cos i}.$$

$$\therefore\ b = -a\frac{\tan(i-r)}{\tan(i+r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Equations (19) and (20) are identical with Fresnel's equations for the amplitude of the refracted and reflected light, *when the vibrations are in the plane of incidence,* except that the value of $c$ is increased in the ratio $\sin i/\sin r = \mu$. According to (20), $b = 0$ when $\tan(i + r) = \infty$, and $i + r = \pi/2$. Since we know from experiment that light polarised perpendicular to the plane of incidence is totally transmitted for an angle of incidence $i$, given by the equation $i + r = \pi/2$, MacCullagh's results can only be brought into conformity with Brewster's law (p. 475) by assuming that *the vibrations of polarised light are performed in the plane of polarisation,* instead of perpendicular to it, as assumed by Fresnel. As we have seen, it is practically certain that Fresnel's

assumption is correct, so that MacCullagh's theory must be abandoned. On the other hand, Fresnel's theory cannot be considered sound, since it implies a separation of the ether perpendicular to the surface.

**General Summary of our Present Knowledge.**—Green has fully worked out the problem of reflection and refraction at the interface of two elastic solids, when the elasticities are equal and the densities differ. He found that there would be no angle of complete polarisation. Since experiment shows, that in substances of which perfectly plane and smooth surfaces can be obtained, complete polarisation occurs (p. 476), it follows that Green's theory falls to the ground. The direction of vibration, according to Green's theory, is perpendicular to the plane of polarisation, as proved by experiment.

Lord Rayleigh has worked out an elastic solid theory, according to which the densities of the two media are the same, while their elasticities differ. He found that in this case there would be *two* polarising angles, which is contrary to experience.

Thus, the reflection and refraction of light by isotropic media cannot be satisfactorily explained in terms of the properties of ordinary elastic solids. Even when the reactions of the molecules are taken into account, as in Sellmeier's theory (p. 375), the difficulties are in no way removed; as pointed out on p. 283, an increase in the effective density of the medium is produced by these reactions, and Fresnel's energy equation still holds.

On the other hand, Fresnel's *results* are in very close agreement with experimental facts; consequently we may conclude that similar results may possibly be obtainable, this time in a satisfactory manner, from some other theory of the nature of light, or of the medium by means of which light-vibrations are transmitted.

## ÆOLOTROPIC MEDIA

**General Conditions.**—Any physical agency, when acting on a crystal, will generally produce different effects in different directions in the crystal. Thus, the coefficient of linear expansion of a crystal will have different values in different directions, and a similar variation usually occurs in respect to

thermal conductivity, hardness, cleavage, elasticity, etc. We have also seen that the refraction of light by a crystal presents characteristics which vary with the direction of vibration.

Fresnel assumed that, when a plane light-wave passes through a crystal, the direction of vibration of an ether particle is always parallel to the wave-front, and perpendicular to the direction in which the wave is transmitted. He further assumed that the restoring force called into play by the displacement of an ether particle, depends on the absolute displacement of that particle. As we have seen (p. 274), according to a correct elastic solid theory, the restoring force depends on the displacement of a particle *relatively to surrounding particles*, so that Fresnel's assumption in this respect is defective. Indeed, on this assumption, it is difficult to see how vibrations could be handed on from particle to particle, so as to constitute progressive waves; any mechanical connection between neighbouring particles would necessitate reactions depending in some manner on their relative displacements. However, if we accept Fresnel's assumption, the coefficient of elasticity of the ether must be measured by the restoring force called into play by a unit linear displacement of an ether particle. The velocity of wave propagation being assumed to be equal to the square root of the ratio of the elasticity to the density of the ether, from a loose analogy with the reasoning given on p. 271, it follows that, if the restoring force on an ether particle varies with the direction of displacement, the velocity of wave transmission will vary with the direction of vibration in the wave-front. It should be noted that Fresnel's theory contemplates only motions of the ether; the reactions of the matter molecules are supposed *merely to modify the elasticity of the ether*, so that ethereal displacements in different directions call into play different restoring forces, while *the density of the ether is unaffected by the presence of matter molecules;* this is, of course, quite inconsistent with his assumptions made to explain the reflection and refraction of light by isotropic media.

**Principal Axes of Elasticity.**—Fresnel next assumed that, within a crystal, there are three directions, each one being at right angles to the plane containing the other two, characterised by the property that in either of them the displacement of an ether particle, and the restoring force called into play, are in

the same straight line. The restoring forces corresponding to unit displacements along these directions are defined as the **principal** (optical) **elasticities** of the crystal.

Drawing three straight lines parallel to these directions, so as to intersect in a single point, we obtain a system of three rectangular axes of co-ordinates. We shall term these the axes of $x$, $y$, and $z$, respectively. Let the principal elasticities along $x$, $y$, and $z$, be respectively equal to $a^2$, $b^2$, and $c^2$.

Consider an ether particle, initially situated at the origin, but now displaced through unit distance in a straight line, inclined to the axes of $x$, $y$, and $z$, at angles of which the cosines are equal to $l_1$, $l_2$, $l_3$. Then, since the distance of the particle from the origin is equal to unity, the rectangular co-ordinates, $x$, $y$, $z$, of the particle will be given by—

$$x = l_1 \times 1 = l_1, \qquad y = l_2 \times 1 = l_2, \qquad z = l_3 \times 1 = l_3.$$

Thus, the given unit displacement can be resolved into three components, respectively equal to $l_1$, $l_2$, $l_3$, in directions parallel to the axes of $x$, $y$, and $z$.

The displacement $l_1$, along the axis of $x$, will call into play a restoring force equal to $a^2l_1$, since $a^2$ is the restoring force corresponding to unit displacement in that direction. Similarly, the component restoring forces along the $y$ and $z$ axes will be equal to $b^2l_2$ and $c^2l_3$ respectively. Since the resultant restoring force on the particle is equivalent to the three components $a^2l_1$, $b^2l_2$, and $c^2l_3$, it follows that the direction cosines of this resultant will be proportional to $a^2l_1$, $b^2l_2$, and $c^2l_3$. As a consequence, the resultant restoring force will not, as a general rule, be in the same line as the displacement, so that it will not tend to bring the displaced particle back to its position of equilibrium. The only exceptions occur when the displacement is along one or other of the axes.

**Wave Propagation in a Crystal.**—Let us suppose that a plane wave is transmitted through the crystal. All particles in the wave-front will, at a given instant, be displaced in the same direction; and in order that the wave should be transmitted without alteration, it is necessary that these displacements should give rise to exactly similar displacements in the new wave-front. But if the restoring force on a particle is not in the same straight line as the displacement, the reaction of the

particle will produce in the new wave-front displacements which are not parallel to those in the old wave-front. Thus, the character of the wave would alter during transmission.

It is found that in a given wave-front there are always two directions, at right angles to each other, such that a displacement in either will give rise to a restoring force *in the same plane as the displacement and the wave normal.* In these cases the restoring force only comprises a component parallel to the displacement, and another perpendicular to the wave-front. Assuming that vibrations can only be performed in a direction perpendicular to that of wave propagation, and therefore in the wave-front, the component of the restoring force perpendicular to the wave-front can produce no effect on the direction of vibration, so that, with respect to vibrations performed in the directions mentioned, the component of the restoring force resolved parallel to the displacement is alone operative in propagating the waves. Thus, **in a given wave-front, there are always two directions, at right angles to each other, such that vibrations along these can be transmitted without alteration.**

Thus, we must suppose that a plane wave of unpolarised light, after entering a crystal at normal incidence, at first passes through a transition stage in which the vibrations are continually altering their directions in the wave-front. After penetrating the crystal to a very small distance, the vibrations entirely settle down to two directions in the wave-front at right angles to each other. No energy has been lost, and that of the incident vibrations will be equally divided between the two sets of vibrations transmitted through the crystal. Thus, the transmitted light becomes polarised in two planes at right angles to each other.

**Velocity of Wave Transmission.**—The wave-front is always perpendicular to the direction in which it is transmitted. The two directions, perpendicular to the direction of wave transmission, and to each other, in which vibrations can be permanently executed, are determined by reasoning similar to that explained above. The restoring forces corresponding to unit displacements along these directions will generally be different, so that the elasticity of the ether for vibrations in these directions will also be different. Since the velocity of wave transmission varies as the square root of the elasticity of the

ether, the vibrations in the two directions defined above will produce two separate waves travelling in the same direction with different velocities.

**Wave Surface.**—Let us suppose that, at a given instant, a great number of plane waves are passing, in different directions, through a point in a crystal. Corresponding to each direction of wave transmission there will in general be two waves travelling with different velocities, and these will of course traverse different distances in the same time. If we draw, through the given point, a great number of symmetrically distributed straight lines, to represent the directions in which the various waves are transmitted, we can mark off on each the two distances through which waves will be transmitted in the given time in that direction. Let a plane be drawn through each point so found, so as to be perpendicular to the line on which the point is situated. These planes will represent the various wave-fronts. The intersections of these planes will envelop two curved surfaces, which constitute the two sheets of Fresnel's wave surface.

The determination of the equation to this surface constitutes a problem in Analytical Geometry of Three Dimensions.[1] Here we shall content ourselves with the simpler problem of finding the intersections of the wave surface by the three planes, containing the three axes, taken two by two. The mathematical difficulties of the problem are thus greatly diminished, while most of the important properties of the surface are rendered evident.

We shall suppose that the three principal elasticities of the crystal, along the axes of $x$, $y$, and $z$, are respectively equal to $a^2$, $b^2$, and $c^2$, where $a^2 > b^2 > c^2$.

**Section of the Wave Surface by the Plane of $xz$.**—Let OX, OZ (Fig. 276), represent the axes of $x$ and $z$, so that the axis of $y$ must be imagined to pass through O perpendicular to the plane of the paper. Let BB be the trace of a plane wave-front perpendicular to the plane of the paper. Thus N′ON, the normal to the wave-front, will lie in the plane of the paper.

Now, it is obvious that displacements perpendicular to the plane of the paper will be parallel to the axis of $y$, and thus will call forth restoring forces parallel to their own directions. Thus,

1 Its full solution can be found in Preston's *Theory of Light*, p. 268.

vibrations in this direction can be transmitted without change through the medium (p. 530). Displacements along OB, in the plane of $xz$, will possess no component along the axis of $y$, so that the resultant restoring force will be equivalent to the restoring forces called into play by the component displacements along OX and OZ. Since these component forces lie in the plane of the paper, their resultant must also do so, and it will therefore lie in the same plane as the displacement (along OB) and the normal to the wave, NON′. Thus, displacements along any line, OB, in the plane of the paper can be transmitted without change (p. 530).

FIG. 276.—Illustrates Wave Propagation in a Crystal.

VIBRATIONS PERPENDICULAR TO THE PLANE OF $xz$.—Unit displacement perpendicular to the plane of $xz$ will call forth a restoring force parallel to the axis of $y$, equal to $b^2$. Thus $b^2$ is the elasticity of the ether for vibrations perpendicular to the plane of $xz$, and a plane wave vibrating in this direction will travel along ON with a velocity equal to $\sqrt{b^2/\rho}$, where $\rho$ is the density of the ether. Let $b^2/\rho = \beta^2$. Then a plane wave vibrating perpendicular to the plane of $xz$ will travel a distance equal to $\beta$ in one second. Since this result is independent of the direction of the normal N′ON, all waves of the class considered will travel through equal distances in a second, and their traces will give a number of straight lines enveloping a circle, with O as centre, and radius $= \beta$. This, then, is the section of the wave surface corresponding to waves vibrating parallel to the axis of $y$ (Fig. 278).

VIBRATIONS IN THE PLANE OF $xz$.—Let ON, the normal to the wave-front, make an angle $\theta$ with the axis OX. Then, since OB is perpendicular to ON, $\angle$ BOX′ $= (\pi/2) - \theta$. Thus, unit

displacement along OB will be equivalent to the components, $1 \times \cos$ BOX′ $= \cos \{(\pi/2) - \theta\} = \sin\theta$ along OX′, and $1 \times \cos$ BOZ $= \cos\theta$, along OZ. The corresponding restoring forces will be equal to $a^2 \sin\theta$ and $c^2 \cos\theta$, along OX and OZ respectively. According to Fresnel's hypothesis, *it is only the component restoring force resolved along the direction of displacement which is operative in wave transmission.* We must therefore resolve the forces acting along OX and OZ in the direction of OB, and add the results. We thus get $(a^2 \sin^2\theta + c^2 \cos^2\theta)$ as the effective restoring force for unit displacement along OB. This gives us the elasticity of the ether for displacements along OB, and the corresponding velocity of transmission is equal to $\sqrt{(a^2 \sin^2\theta + c^2 \cos^2\theta)/\rho}$. Let $a^2/\rho = a^2$, whilst $c^2/\rho = \gamma^2$. Then a plane wave vibrating in the plane of $xz$ (the wave normal being in the plane of $xz$, and inclined to the axis of $x$ at an angle $\theta$) will be transmitted with a velocity equal to $\sqrt{(a^2 \sin^2\theta + \gamma^2 \cos^2\theta)}$.

On ON mark off a point N at a distance equal to $\sqrt{(a^2 \sin^2\theta + \gamma^2 \cos^2\theta)}$. Through N draw the straight line DNE perpendicular to ON. Then the plane wave (vibrating in the plane of $xz$), of which BB is the trace, will, after one second, occupy a position such that DNE is its trace.

FIG. 277.—Waves vibrating in the Plane of $xz$.

If we now, from O (Fig. 277), draw a number of straight lines, representing various directions of wave transmission in the plane of $xz$, and if, through $N_0$, $N_1$, $N_2$ . . . , points found in the manner just described, we draw perpendiculars to represent the traces of the corresponding wave fronts, the intersections of these perpendiculars will give us a curve which is the section

of the wave surface corresponding to waves vibrating parallel to the plane of $xz$ (Fig. 277).

In Fig. 277 the traces of the wave fronts, corresponding to ten different directions of wave propagation, are given. It is seen that these envelop an oval curve. As the angle between successive wave normals (such as $N_2ON_1$) is diminished, the curve becomes more exactly defined. It will be seen that each trace of a plane wave front passes through two points on the curve, and as the number of traces is increased, the distance between these two points diminishes. Consequently, the traces of the plane wave fronts are tangents to the curve, **and the plane wave fronts themselves are tangent planes to the wave surface.**

We can, moreover, obtain the *equation* to this section in a comparatively simple manner. Let DF (Fig. 276) represent the trace of a wave front of which the normal, represented by OM, is very nearly coincident with ON. DE and DF intersect in D, so that D must be a point on the wave surface. We must determine the equation of all such points as D, obtained by the construction described above.

It can be proved[1] that if the perpendicular distance from the origin to a straight line is $p$, and this perpendicular makes an angle, $\theta$, with the axis of $x$, then the equation to the straight line referred to the axes of $x$, $z$, is given by $x\cos\theta + z\sin\theta = p$.

Since $p = \text{ON}$ (Fig. 276) $= \sqrt{(\alpha^2\sin^2\theta + \gamma^2\cos^2\theta)}$, the equation to the straight line DNE is given by—

$$x\cos\theta + z\sin\theta = (\alpha^2\sin^2\theta + \gamma^2\cos^2\theta)^{\frac{1}{2}}. \quad . \quad . \quad . \qquad (1)$$

$$\therefore\; x^2\cos^2\theta + 2xz\cos\theta\sin\theta + z^2\sin^2\theta = \alpha^2\sin^2\theta + \gamma^2\cos^2\theta.$$

$$\therefore\; (z^2 - \alpha^2)\sin^2\theta + 2xz\sin\theta\cos\theta + (x^2 - \gamma^2)\cos^2\theta = 0.$$

Dividing both sides by $(z^2 - \alpha^2)\cos^2\theta$, we obtain—

$$\tan^2\theta + \frac{2xz}{z^2 - \alpha^2}\tan\theta + \frac{x^2 - \gamma^2}{z^2 - \alpha^2} = 0. \quad . \quad . \quad . \quad . \qquad (2)$$

When definite values are given to $x$ and $z$, (2) becomes a quadratic equation in $\tan\theta$, which will generally be satisfied by two separate values of $\theta$. Thus, the point D (Fig. 276), will be the point of intersection of two lines, such as DNE, DMF, of which the perpendiculars from the origin are inclined to the axis of $x$ at the angles $\theta_1 = \angle\,\text{NOX}$,

[1] *The Elements of Co-ordinate Geometry*, by S. L. Loney (Macmillan), p. 33.

and $\theta_2 = \angle$ MOX. If the angle MON is infinitesimally small, these two angles should become equal. Thus, if we find the connection between $x$ and $z$, in order that (2) should give two equal values for $\tan\theta$, we shall obtain the condition that the point $(x, z)$ lies on the curve enveloped by the variable straight line given by (1). From (2)—

$$\tan^2\theta + \frac{2xz}{z^2 - a^2}\tan\theta + \left(\frac{xz}{z^2 - a^2}\right)^2 = \left(\frac{xz}{z^2 - a^2}\right)^2 - \frac{x^2 - \gamma^2}{z^2 - a^2}$$
$$= \frac{x^2z^2 - (z^2 - a^2)(x^2 - \gamma^2)}{(z^2 - a^2)^2}.$$
$$\therefore\ \tan\theta = -\frac{xz}{z^2 - a^2} \pm \sqrt{\frac{x^2z^2 - (z^2 - a^2)(x^2 - \gamma^2)}{(z^2 - a^2)^2}}.$$

In order that this should give two equal values for $\tan\theta$, the quantity under the radical sign must be equal to zero. Thus—

$$x^2z^2 - (z^2 - a^2)(x^2 - \gamma^2) = x^2z^2 - x^2z^2 + z^2\gamma^2 + x^2a^2 - a^2\gamma^2 = 0.$$
$$\therefore\ \frac{x^2}{\gamma^2} + \frac{z^2}{a^2} = 1.$$

It will be seen on reference to a work on coordinate geometry,[1] that this equation represents an ellipse, the principal semi-axes of which lie in the axes of reference; that in the axis of $x$ having a length equal to $\gamma$, while that in the axis of $z$ has a length equal to $a$. This ellipse is the section of the wave surface by the plane of $xz$, corresponding to vibrations in the plane of $xz$.

The complete section of the wave surface by the plane of $xz$ is thus a circle, of radius $\beta$, and an ellipse, of which the semi-axes lie in the axes of $x$ and $z$, and are respectively equal to $\gamma$ and $a$. This section is shown in Fig. 278. It will be noticed that the ellipse and circle intersect in four points, in two straight lines. This is due to the circumstance that the plane of $xz$ contains the axes of greatest and least elasticity of the crystal. We shall afterwards find that it has important consequences.

FIG. 278.—Section of Wave Surface by Plane of $xz$.

**Section of the Wave Surface by the Plane of $xy$.**—To find this, we proceed exactly as in the case of the section by the plane of $xz$. In Fig. 276 we need only imagine that the line OZ

[1] Loney, p. 226.

represents the axis of $y$, while the axis of $z$ is perpendicular to the paper.

All vibrations perpendicular to the plane of $xy$, will be parallel to the axis of $z$, and for these the elasticity of the ether will be equal to $c^2$, so that all wave-fronts perpendicular to the plane of $xy$, the vibrations being parallel to the axis of $z$, will be transmitted with a velocity equal to $\sqrt{c^2/\rho} = \gamma$. Thus, a circle with radius $\gamma$ will be one part of the section of the wave surface by the plane of $xy$ (Fig. 279).

Turning our attention to vibrations in the plane of $xy$, let the wave normal make an angle, $\theta$, with the axis of $x$. Then the elasticity of the ether for vibrations in the corresponding wave-front will be equal to $a^2 \sin^2\theta + b^2 \cos^2\theta$, and the velocity of wave propagation will be equal to $\sqrt{(\alpha^2 \sin^2\theta + \beta^2 \cos^2\theta)}$. The equation to the line which is the trace on the plane of $xy$ of the wave-front one second after it passed through the origin, will be given by—

$$x \cos\theta + y \sin\theta = (\alpha^2 \sin^2\theta + \beta^2 \cos^2\theta)^{\frac{1}{2}},$$

and, proceeding in the manner previously explained, we find that, if various lines for different values of $\theta$ are drawn, they will envelop the ellipse, given by the equation—

$$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = 1.$$

Thus, the section of the wave surface by the plane of $xy$ consists of an ellipse, of which the semi-axes agree with the reference axes of $x$ and $y$, and have the values $\beta$ and $\alpha$ respectively, together with a circle of radius $\gamma$. In this case the circle lies entirely within the ellipse (Fig. 279).

FIG. 279.—Section of Wave Surface by Plane of $xy$.

**Section of the Wave Surface by the Plane of $zy$.**—The student should have no difficulty in proving that this will consist of an ellipse, of which the semi-axes agree with the reference axes of $z$ and $y$, and have values respectively equal to $\beta$ and $\gamma$; together with a circle of radius $\alpha$. In this case the ellipse is entirely enclosed by the circle (Fig. 280).

**General Form of Fresnel's Wave Surface.**—This can be seen from Fig. 281, which gives a perspective view of the

sections by the three reference planes. Fig. 282 shows the wave surface with the front half of the outer shell removed.

**Principal Refractive Indices of a Crystal.**—From the reasoning already employed, it follows that all plane waves, vibrating parallel to the axis of $x$, will possess a velocity equal to $a$. Let $v_0$ be the velocity of light *in vacuo.* Then the refractive index of the crystal for waves vibrating parallel to the axis of $x$ will be equal to $v_0/a$. Similarly $v_0/\beta$ and $v_0/\gamma$ will be the refractive indices for waves vibrating parallel to the axes of $y$ and $z$, respectively. The quantities $v_0/a$, $v_0/\beta$, and $v_0/\gamma$ are termed the **principal refractive indices** of the crystal.

FIG. 280.—Section of Wave Surface by Plane of $yz$.

**Direction of the Ray.**—Let ED, FD (Fig. 276), represent two nearly parallel plane waves which passed simultaneously through the origin O. Since the medium is æolotropic, the distances travelled by these waves will be unequal; in the figure OM > ON. In the immediate neighbourhood of D the two waves reinforce each other; thus, D will be a position of maximum disturbance. At other points the displacements due to these waves will be neutralised by other waves in different phases. Hence the disturbance produced by the waves ED and FD will only be sensible at D. It is obvious that the disturbance at D has previously passed along the line OD; thus OD **is the ray corresponding to**

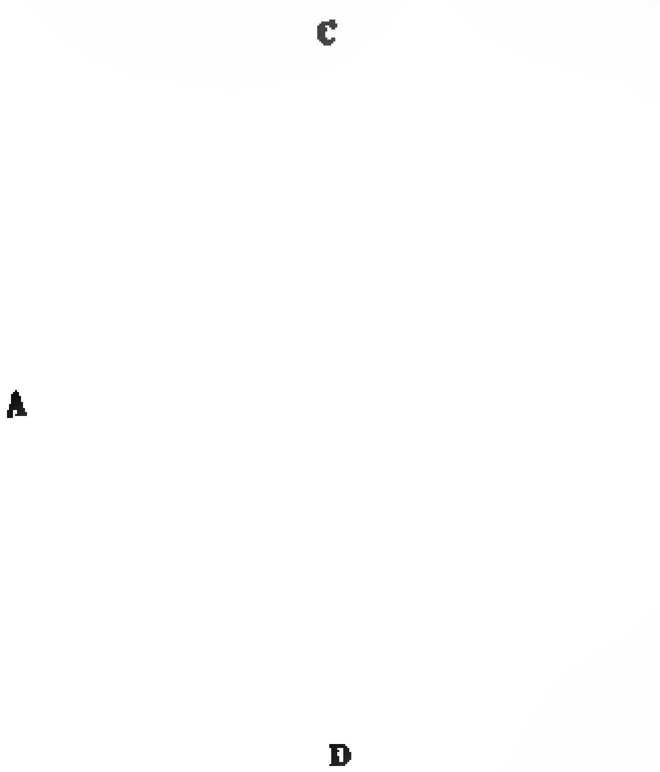

FIG. 281.—Sections of Fresnel's Wave Surface.

**plane waves sensibly parallel to** ED. It must be noticed that in general the ray is not perpendicular to the wave-front.

**Connection between Ray and Vibration Directions.**—We can now deduce an important law, connecting the direction of vibration in a plane wave front with the direction of the ray. Since DNO (Fig. 276) is a right-angled triangle, DN is the

FIG. 282.—Fresnel's Wave Surface, with front half of Outer Shell removed.

projection of OD on the wave front. But the vibrations corresponding to the wave front of which DNE is a section, are in the plane of the paper, and are therefore along DN. Thus, **if we project the ray on the corresponding wave front, we obtain the direction of vibrations in that ray.**

**Optic Axes.**—In the plane of *xz*, which contains the axes of greatest and least elasticities of the crystal, there are two

directions of wave propagation such that the velocity of transmission is the same, whatever may be the direction of vibration in the wave front.

From the reasoning on pp. 532–3, we see that vibrations *perpendicular to the plane of $xz$* are transmitted with a uniform velocity equal to $\beta$, whatever may be the direction of wave propagation. Vibrations *in the plane of $xz$* are transmitted with a velocity equal to—

$$\sqrt{(a^2 \sin^2 \theta + \gamma^2 \cos^2 \theta)} = \sqrt{(\gamma^2 + (a^2 - \gamma^2) \sin^2 \theta)},$$

where $\theta$ is the inclination of the wave normal to the axis of $x$. Now, $\sqrt{(\gamma^2 + (a^2 - \gamma^2) \sin^2 \theta)}$ will increase from $\gamma$ to $a$ as $\theta$ is increased from 0 to $\pi/2$, and since $a > \beta > \gamma$ (p. 531), it follows that for some value of $\theta$ between these limits we shall have—

$$[\gamma^2 + (a^2 - \gamma^2) \sin^2 \theta]^{\frac{1}{2}} = \beta.$$

For this value of $\theta$, vibrations in, and perpendicular to, the plane of $xz$ will be transmitted with equal velocities, and it can be proved that in this case all vibrations, whatsoever may be their directions in the wave front, can be transmitted without alteration with one uniform velocity.

To find the values of $\theta$ for which this occurs, we have—

$$\gamma^2 + (a^2 - \gamma^2) \sin^2 \theta = \beta^2 ; \therefore (a^2 - \gamma^2) \sin^2 \theta = \beta^2 - \gamma^2,$$

and

$$\sin \theta = \pm \sqrt{\frac{\beta^2 - \gamma^2}{a^2 - \gamma^2}}.$$

The meaning of this can be seen on reference to Fig. 283. As already explained (p. 534), the trace of a plane wave front on the plane of $xz$ will be a tangent to the section of the corresponding sheet of the wave surface. As a general rule, tangents to the ellipse and circle (Fig. 283), which are perpendicular to one common direction, will not coincide, so that there are generally two different velocities of wave transmission in a given direction, corresponding to vibrations in, and perpendicular to, the plane of $xz$. But there are two directions of transmission, equally inclined to the axis of OX, corresponding to which a tangent to the circle is also a tangent to the ellipse. These directions are determined by the above values of sin $\theta$. Since $\theta$ is the inclination to the axis of $x$, of the perpendicular

from the origin on to the tangent, it follows that the two values of $\theta$ correspond to the angles NOX and N′OX, where N′OX′ = NOX.

**The two directions in a crystal, along which plane waves may be transmitted with one uniform velocity, whatever may be the direction of the vibrations in the wave front, are termed the optic axes of the crystal.**

Fig. 283.—Optic Axes of a Crystal.

Thus, in Fig. 283, which represents the section of the wave surface of a crystal by the plane containing OZ and OX, the axes of greatest and least elasticity, the lines ON and ON′ indicate the directions of the optic axes.

**Internal Conical Refraction.**—Although plane waves travel along the optic axes of a crystal with one uniform velocity, whatever may be the direction of vibration in the wave front, the corresponding *rays* will pursue very different paths. In Fig. 283, OM represents the direction of the ray corresponding to vibrations in the plane of $xz$ (*i.e.* along MN), while ON represents the ray corresponding to vibrations perpendicular to the plane of $xz$ (*i.e.* perpendicular to the plane of the paper).

The *tangent line* NM (Fig. 283) touches the ellipse and circle at the *points* M and N. Sir William Hamilton proved, however, that the *tangent plane*, of which NM is the trace, touches the wave surface along a *circle* of which NM is a diameter. The points P, P′ (Fig. 281), are the centres of small conical depressions in the wave surface, and the tangent planes of which NM, N′M′ are the traces, cover these up, making contact with the surface along a circle (compare Fig. 282). But, corresponding to each point of contact between the tangent plane and the surface, there will be a definite ray of light; consequently there will be a *hollow cone of rays*, diverging from O, and passing through the

circle of contact. Since all of these rays correspond to coincident plane waves, if they are refracted into the air from the crystal they will be rendered parallel, and will produce a hollow cylinder of rays (Fig. 284).

At Sir William Hamilton's invitation, Dr. Lloyd made an experimental inquiry into the truth of this conclusion. He chose a crystal of aragonite, since in that case the angle of the internal cone is greater than in most other crystals, and its principal indices of refraction had previously been carefully measured by Rudberg. The crystal was cut with its two opposite faces perpendicular to the axis of least elasticity, and a very narrow linear pencil of light, passing through the apertures in two screens (Fig. 285), was refracted through the crystal in the plane containing the optic axes of the crystals. By moving the screen placed in contact with the upper surface of the crystal, the angle of incidence was varied. The refracted rays, after passing through the crystal, fell on a screen EF, and in general produced two white spots; but at a certain angle of incidence these spots enlarged so as to form a luminous ring with a dark centre, so that Sir William Hamilton's prediction was fully verified. The angle of incidence at which this occurred was found by receiving the reflected ray OK on a screen, so that the value of the angle was obviously equal to half the angle SOK. The angle so found was

FIG. 284.—Internal Conical Refraction.

FIG. 285.—Method of observing Internal Conical Refraction.

in perfect agreement with theory, as was also the vertical angle of the internal cone of rays.

Mr. W. B. Croft has obtained a photograph by allowing the hollow cylinder of rays emerging from the crystal to fall on a photographic plate. The screen placed in contact with the surface of the crystal was perforated with five small apertures, so that five narrow linear pencils were refracted through the crystal. The central aperture was in such a position that the pencil admitted by it formed the internal cone of rays; corresponding to this aperture we have the central bright ring (Fig. 286). The pencils admitted by the other apertures were inclined to the direction in which internal conical refraction occurs, so that the emergent pencils were not cylindrical. The manner in which the two images, due to double refraction, are related to the single circular image due to internal conical refraction, is strikingly shown.

**Polarisation of Cone of Rays.**—It has already been remarked that the vibrations corresponding to the rays OM and ON (Fig. 287) are performed in directions at right angles to each other. It can easily be seen that any two rays, passing through points at opposite ends of a diameter to the circle of contact of the tangent plane anc will be polarised at righ other.

Fig. 286.—Results of Internal Conical Refraction.

For, let ON (Fig. 287) c (Fig. 283), while NPQ re spective view of the cir Then ON is perpendicula the circle NPQ. Let P points at opposite ends of a diameter PQ to the Then the projection of OP on the plane of N tangent plane to the surface at P) will give vibration at P; this is seen to be equal to PN vibrations in the ray passing through Q will be in the direction QN. But PN and QN are at right angles, since $\angle$PNQ is subtended at a point N, on the circumference of the circle PNQ, by the diameter PQ.

**Axes of Single Ray Velocity.**—The ray velocity must not be confused with the wave velocity. Corresponding to waves transmitted with uniform velocity in the direction of the optic axes of a crystal, there is a number of different rays, of which the

velocities vary with the directions of vibration in the wave front. Thus, in a given time, the ray corresponding to vibrations in the plane of $xz$ (Fig. 283) will travel through the distance OM; while the ray corresponding to vibrations perpendicular to the plane of $xz$ will, in the same time, travel through the distance ON.

FIG. 287.—Illustrates the Polarisation of the Internal Cone of Rays.

On the other hand, the lines OP, OP′ (Fig. 288), are termed **axes of single ray velocity.** These lines cut the wave surface at the apex of the conical depression before mentioned. At this point two tangent planes, perpendicular to the plane of $xz$, can be drawn to the wave surface; the traces of these planes are shown in Fig. 288, as tangents to the section of the wave surface. But P is the apex of a *conical depression.* Thus, the section of the wave surface by a plane passing through OP, but inclined to the plane of $xz$ at any angle whatever, will permit of two tangent lines being drawn at P, and these lines will be the traces of two planes perpendicular to the plane of section. Thus, at P an indefinitely large number of tangent planes can be drawn, and these planes will envelop a cone, termed the *tangent cone*, at the point P. Consequently, the ray OP will correspond to an indefinitely large number of plane waves which travel within the crystal with different wave velocities, but one single ray velocity.

FIG. 288.—Tangents to Section of Wave Surface.

The direction of the axes of single ray velocity can easily be found. For, if $x$, $y$, are the co-ordinates of the point P, and $\angle$ XOP = $\phi$, we have—

$$\sin \phi = \frac{z}{(x^2 + z^2)^{\frac{1}{2}}},$$

Further, since P lies on the circle ($x^2 + z^2 = \beta^2$), and also on the ellipse

$$\frac{x^2}{\gamma^2} + \frac{z^2}{\alpha^2} = 1,$$

we have, substituting for $x$,

$$\frac{\beta^2 - z^2}{\gamma^2} + \frac{z^2}{\alpha^2} = 1, \therefore z^2\left(\frac{1}{\alpha^2} - \frac{1}{\gamma^2}\right) = 1 - \frac{\beta^2}{\gamma^2};$$

$$\therefore z = \pm\, \alpha \sqrt{\frac{\beta^2 - \gamma^2}{\alpha^2 - \gamma^2}},$$

and

$$\sin\phi = \frac{z}{\beta} = \pm\, \frac{\alpha}{\beta} \sqrt{\frac{\beta^2 - \gamma^2}{\alpha^2 - \gamma^2}}.$$

**External Conical Refraction.**—When a ray, after traversing a crystal, is refracted into the air, the direction of the ray in the air is determined by the position of the tangent plane to the wave surface at the point cut by the ray in the crystal. Now, at the point on the wave surface cut by the ray OP, there is an indefinitely large number of tangent planes enveloping a cone; therefore the ray OP, after emerging from the crystal, will separate itself into an indefinitely large number of rays forming a hollow cone (Fig. 289.)

FIG 289.—Illustrates External Conical Refraction.

This result, which was predicted by Sir William Hamilton, was verified experimentally by Dr. Lloyd. A conical pencil of light was focussed on a point, O (Fig 290), on the upper face of the crystal of aragonite already mentioned. Two diaphragms were placed on opposite faces of the crystal so that the line joining the small apertures they respectively possessed coincided with the axis of single ray velocity in the crystal. Out of the whole cone of rays falling on O, the rays corresponding to

a certain hollow cone were refracted so as to coincide with the axis of single ray velocity, and finally produced a hollow cone of rays on emergence from the lower face of the crystal. Thus, an eye placed beneath the crystal, saw a brilliant annulus of light, and Sir William Hamilton's prediction was entirely fulfilled.

**Relation between the Planes of Polarisation and the Optic Axes.**—The plane of $yz$ passes through the axis of $z$, which is the bisector of the angle between the optic axes. All waves of which the normals lie in the plane of $yz$, are polarised either in, or perpendicular to, that plane (p. 536). It can be proved that the planes of polarisation for any wave whatever may be found in the following simple manner. From the centre of the wave surface draw a straight line parallel to the wave normal. Through this line draw two planes, each passing through one of the optic axes. Then **the planes of polarisation of the two waves with the given normal, bisect the internal and external angles between the two planes, each containing the wave normal and one of the optic axes.**

FIG. 290.—Method of observing External Conical Refraction.

We have just seen that this construction will suffice to determine the planes of polarisation of waves of which the normals lie in the plane of $yz$. It will also obviously suffice for waves of which the normals lie in the plane of $xz$, for in this case the vibrations are either in, or perpendicular to, the plane of $xz$ (p. 532), and this plane passes through the optic axes, and therefore contains their bisector.

It may also be shown that this construction will suffice for waves of which the normals lie in the plane of $xy$. For let Fig. 291 represent the section of the wave surface by the plane of $xy$. Let AB, ND, be the traces of plane waves, of which the normals are parallel to ONM. The optic axes lie in a plane through XOX′, perpendicular to the plane of the paper, and the inclination of one axis to OX is equal to the inclination of the other axis to OX′. Imagine planes to be drawn through ON and

the optic axes; then the external angle between these planes will be bisected by the plane of the paper, and the internal angle will be bisected by a plane through ON, perpendicular to the plane of the paper. The vibrations in the wave AB are parallel to AB, and the plane of polarisation of this wave is therefore perpendicular to the plane of the paper. The vibrations in the wave ND are perpendicular to the plane of the paper, and this wave is therefore polarised in the plane of the paper.

FIG. 291.—Illustrates the relation between the Planes of Polarisation and the Optic Axes.

**Dispersion of the Optic Axes.**—The elasticities of the ether must be supposed, according to Fresnel's theory, to vary not only with the direction of vibration, but also with the wave-length of the light transmitted. As a consequence, there will be a separate wave surface for each wave-length of light, and, in particular, the optic axes of a crystal will be different for waves of different lengths.

**Uniaxal Crystals.**—If any two of the principal wave velocities, $\alpha, \beta, \gamma$, become equal, Fresnel's wave surface degenerates into a sphere and an ellipsoid of revolution, as assumed by Huyghens. Thus, let $\beta = \alpha$. Then the sections by the planes of $xz$ and $yz$ are similar, each consisting of an ellipse, with major semi-axis equal to $\alpha$, in the direction of the axis of $z$, and with minor semi-axis equal to $\gamma$, in the direction of $x$ or $y$; together with a circle of radius equal to $\alpha$. The section by the plane of $xy$ degenerates into two circles, of radii equal to $\alpha$ and $\gamma$ respectively (Fig. 279, p. 536). The inner and outer sheets of the wave surface touch at their intersection with the axis of $z$. Thus, in this case the axis of $z$ becomes the optic axis, and also the axis of single ray velocity of the crystal. The angles $\theta$ and $\phi$ (pp. 539 and 543) each become equal to $\frac{\pi}{2}$. This wave surface corresponds to a negative uniaxal crystal, such as Iceland spar. The ellipsoid is entirely within the sphere (Fig. 261, p. 491).

If we examine the case where $\gamma = \beta$, we find that this corresponds to a positive uniaxal crystal, such as quartz, in which the sphere is entirely within the ellipsoid (Fig. 260, p. 491).

Returning to the consideration of negative uniaxal crystals, it will be seen, on comparison with the method of determining the sections of Fresnel's wave surface, that the outer circle, of radius $a$, in the section by the plane of $xy$, corresponds to vibrations in the plane of $xy$. This circle is obviously the section of a sphere of radius equal to $a$; and the ray obtained by joining the centre of the wave surface to the point of contact of a tangent plane (p. 492) coincides with the normal, and is the ordinary ray. The corresponding *principal plane* (p. 487), will pass through this point of contact and the optic axis, and will consequently be perpendicular to the plane of $xy$. Thus, the vibrations in the ordinary ray are perpendicular to the principal plane. Since experiment shows that the ordinary ray is polarised *in* the principal plane (p. 487), we see that Fresnel's construction is consistent with the plane of vibration being perpendicular to the plane of polarisation.

**Criticism of Fresnel's Theory.**—As already remarked, Fresnel's assumption, that the restoring force called into play by the displacement of an ether particle is proportional simply to the absolute displacement of that particle, is inconsistent with any connection between neighbouring ether particles, and could not lead to progressive wave propagation. Another point of serious difficulty lies in ignoring the effect of the reaction perpendicular to the wave front (p. 530). This reaction would lead to the production of longitudinal waves.

It may very plausibly be argued that, if these longitudinal waves were formed, they might be unable to affect our eyes as light does; but on leaving the surface of a crystal, longitudinal vibrations would originate transverse waves of the same period, unless the incidence was normal. Since no such effect has ever been observed, Fresnel's theory must be considered defective in this respect also.

On the other hand, the form of the wave surface obtained by Fresnel is in very close agreement with experiment. After an exhaustive experimental examination, Mr. Glazebrook came to the conclusion that the true form of the wave surface in a crystal, though not absolutely in agreement with Fresnel's construction,

is so very nearly so that there can remain no doubt as to its substantial accuracy.

**Green's Theory.**—Green has investigated the true form of the wave surface in an æolotropic elastic medium having three rectangular planes of symmetry. He assumed that the density of the ether is everywhere the same, but that its rigidity varies with the direction of the shearing strain (p. 267). In order to account for the absence of longitudinal waves, he assumed that the compressional elasticity is very great in comparison with the rigidity. He obtained Fresnel's wave surface, but his reasoning led to the conclusion that the vibrations are parallel to the plane of polarisation, instead of perpendicular to it. This shows that the phenomena of double refraction cannot be accounted for on the supposition that the ether in a crystal has the properties of an ordinary elastic solid with æolotropic rigidity. As Lord Rayleigh has shown (p. 527), there would, moreover, in this case be two angles of polarisation by reflection.

**Later Theories.**—Lord Rayleigh has investigated the form of the wave surface in an elastic solid of which the elasticity is the same in all directions, while the effective density varies with the direction of vibration. This would represent the case of an isotropic elastic solid, embedded in which are numerous heavy bodies capable of independent vibrations, the period of vibration varying with the direction of displacement (p. 281). He found that the wave surface differed considerably from Fresnel's, so that this theory must be abandoned.

Lord Kelvin has modified Lord Rayleigh's theory so as to obtain Fresnel's wave surface on correct mechanical principles. One of the great difficulties in these investigations is to account for the absence of longitudinal vibrations. The longitudinal elasticity of an isotropic elastic solid is equal to $(\epsilon + \frac{4}{3}\eta)$, where $\epsilon$ is the compressional elasticity, and $\eta$ is the simple rigidity (p. 269). The longitudinal vibrations will be propagated with a velocity equal to $\sqrt{(\epsilon + \frac{4}{3}\eta)/\rho}$.

Green assumed that $\epsilon$ was infinitely great in comparison with $\eta$, so that the velocity of propagation becomes infinite. Lord Kelvin assumes that $\epsilon + \frac{4}{3}\eta = 0$, so that the velocity of propagation is equal to zero. This leads to the conclusion that $\epsilon$ must be negative, and equal to $- \frac{4}{3}\eta$. As a consequence, a diminution in volume would lead to a decrease in pressure (measured in the positive direction, *i.e.* outwards), or to an increased contractile tension. To obtain an idea of a medium with such properties, consider a closed vessel entirely filled with foam or froth. The surface of each of the small bubbles of which the foam is composed tends to contract, on account of its surface tension; the enclosed air, however, prevents collapse. Suppose that we remove

the air completely. The foam would now at once collapse, but that the outside layer clings to the walls of the vessel. Thus an inward pull is exerted on the walls. This pull would increase in magnitude if contraction occurred, while the energy of the foam, which is equal to the surface energy of all the component bubbles, would at the same time diminish.

Lord Kelvin assumes the ether to be of the nature of such foam. He has termed it the *labile ether*. Any alteration of shape would be resisted by a definite restoring force, so that transverse vibrations could be propagated through it. It would collapse, however, only for the circumstance that it extends through boundless space. Solid bodies, such as the planets, could move freely through it. The ether penetrating matter has its effective density modified by the matter molecules, which are supposed to be capable of independent vibrations. Mr. Glazebrook has shown that on these suppositions we obtain Sellmeier's dispersion formula (p. 375), so that ordinary and anomalous dispersion may be explained. Fresnel's formulæ for reflection and refraction at the interface of isotropic media are also obtained. If the molecules are arranged symmetrically, their vibration periods being different in different directions, so that the effective density of the ether is different for different directions of displacement, the form of the wave surface would agree with that obtained by Fresnel.

Thus Lord Kelvin's theory of a labile ether, in conjunction with Sellmeier's theory of material particles capable of independent vibrations, affords a consistent explanation of reflection, together with both ordinary and double refraction.

## Questions on Chapter XIX

1. Calculate, according to Fresnel's theory, the intensities of the reflected and refracted rays when light falls upon a transparent medium at perpendicular incidence.

2. Describe the method of exhibiting, and give a general explanation of, the phenomena of internal and external conical refraction in a biaxal crystal.

# CHAPTER XX

## COLOURS OF CRYSTALLINE PLATES

**Parallel Rays : Uniaxal Crystal.**—A parallel pencil of plane-polarised light, transmitted normally through a plate of uniaxal crystal (such as calcite) cut perpendicular to the optic axis, suffers no modification during transmission. If, however, the plate is cut parallel to the optic axis, the case is different. The vibrations in the incident polarised light are resolved parallel and perpendicular to the principal plane of the crystal (p. 499), and the component vibrations are transmitted, as extraordinary and ordinary waves, with unequal velocities. On emergence from the crystal the light will be polarised in the original plane **only when the phase difference of the two sets of waves amounts to** $0, 2\pi, 4\pi, 6\pi, \ldots$ &c. In general the light will be elliptically polarised, and in that case it cannot be completely extinguished by an analysing Nicol. Further, the phase difference between the ordinary and extraordinary rays depends on the wave-length of the transmitted light. Consequently, if the incident light is white, the different wave-lengths will be polarised differently on emergence, and on analysing the emergent light with a Nicol, brilliant chromatic effects will generally be produced.

1. When the Incident Light is Monochromatic.—To fix our ideas, let ABC (Fig. 292) be the section of an acute-angled wedge of quartz cut so that the optic axis is perpendicular to the plane of section ABC, and parallel to the thin edge, A. Let the double arrow, D, represent the direction of vibration in the polarised light; for simplicity, we may suppose this direction to be inclined at an angle of 45° to the thin edge of the wedge. Let the incident light be monochromatic, from the red

end of the spectrum. Then the nature of the polarisation of the light emerging from various parts of the wedge is shown in the middle diagram (Fig. 292). The ordinary and extraordinary wave vibrations are respectively performed perpendicular and parallel to the thin edge of the wedge, and the extraordinary wave velocity is less than the ordinary wave velocity. The light transmitted at E will suffer no appreciable modification owing to the extreme smallness of the path through the crystal. As the length of path through the quartz increases, the difference of phase between the ordinary and extraordinary wave vibrations increases, and the emergent light passes through the various

FIG. 292.—Polarisation of Light on Emergence from a Wedge of Quartz.

stages of polarisation represented (compare Fig. 268, p. 501). The unaided eye will, however, be unable to detect any difference between the light emerging from different parts of the wedge, since the sum of the intensities of the ordinary and extraordinary rays is constant (p. 488). If, however, the emergent light is examined through a Nicol, the wedge will generally exhibit alternate bright and dark bands parallel to its thin edge. If the analysing Nicol is adjusted so that it intercepts vibrations parallel to D, the points E and K will be quite dark. At F and H, where the phase differences amount to $\pi/2$ and $3\pi/2$ respectively, and the light is circularly polarised, the intensity of the light transmitted by the analyser will be equal to half that of the incident light. At G, where the phase difference amounts to $\pi$, the vibration is rectilinear and perpendicular to that of the incident light, so that it is entirely transmitted by the analysing Nicol. Thus, there will be a bright band at G, which gradually shades off into complete darkness at E and K. If the wedge were prolonged

toward the right, a number of bands, alternately bright and dark, would be encountered.

If, now, the analysing Nicol is rotated through 90°, so as to transmit vibrations parallel to D, and to intercept vibrations at right angles to that direction, there will be bright bands at E and K, and a dark band at G. Thus, parts of the wedge which were bright in the first position of the Nicol, become dark when the Nicol is rotated through 90°, and *vice versâ*.

When the analyser is adjusted so as to transmit vibrations parallel, or perpendicular, to the thin edge of the wedge, the extraordinary ray is transmitted and the ordinary ray is intercepted, or *vice versâ*. In either of these cases the bands disappear, and the illuminations become uniform.

The phase difference $\phi'$, between the ordinary and extraordinary rays which have traversed the wedge at a point where its thickness is equal to $\delta$, is given by—

$$\phi' = 2\pi \frac{\delta}{\lambda'}(\mu'_e - \mu'_o), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\mu'_o$ and $\mu'_e$ are the ordinary and extraordinary refractive indices of the crystal for the wave-length $\lambda'$ (compare p. 500).

Let us now turn our attention to the phenomena presented when the incident polarised light is from the violet end of the spectrum. The phase difference, $\phi''$, between the ordinary and extraordinary rays after traversing a thickness, $\delta$, of the wedge, will now be given by—

$$\phi'' = 2\pi \frac{\delta}{\lambda'}(\mu''_e - \mu''_o), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\mu''_o$ and $\mu''_e$ are the ordinary and extraordinary refractive indices of quartz for the new wave-length, $\lambda''$. A glance at the table on p. 495 shows that the *difference* between the ordinary and extraordinary refractive indices of quartz is nearly (but not quite) the same for the C (red) and G (blue) Fraunhofer lines. Hence, from (2), the phase difference produced in traversing a given thickness, $\delta$, of quartz is approximately inversely proportional to the wave-length of the transmitted light. Consequently, if we assume the violet light to be of half the wave-length of the red light, the violet light emerging from various parts of the wedge will be characterised by the forms of vibration represented in the lower diagram (Fig. 292). If the analysing Nicol is adjusted to intercept vibrations parallel to D, we shall have dark bands at E, G, and K, and bright bands at F and H. On rotating the

analyser through 90°, the bands previously dark become bright, and *vice versâ*.

If, now, the incident polarised light consists of a mixture of red and violet rays, and the analyser is adjusted to intercept vibrations parallel to D, the points E and K will be dark; the points F and H will be seen by means of red and violet rays, the violet preponderating; and the point G will be seen only by means of red rays. On rotating the analysing Nicol through 90°, the point G will be seen only by means of violet rays.

2. When the Incident Light is White.—If we now suppose the incident light to be white, there will be a separate set of bands for each wave-length, and since the bright bands corresponding to different wave-lengths will be formed at different positions, the tint of the emergent light continually changes as we pass along the wedge. The colours will be most brilliant when the analysing Nicol is arranged so as to intercept vibrations parallel, or perpendicular, to those of the incident light—that is, when the analyser and polariser are parallel, or crossed. When the principal section of the analysing Nicol is either parallel, or perpendicular, to the optic axis of the crystalline wedge, all traces of colour vanish. If the analysing Nicol is adjusted so as to produce chromatic effects, then rotating it through 90° will cause the tint of each point to change to its complementary. This result follows from the circumstance that the colour at any point is due to the interception of certain wave-lengths by the analysing Nicol; rotating the latter through 90° allows the wave-lengths previously intercepted to be transmitted, while those previously transmitted are now intercepted.

We may now suppose the quartz wedge to be replaced by a uniform plate of the crystal cut parallel to the axis. The plate will appear of a uniform colour, similar to that of the part of the wedge which was equal to the plate in thickness. On rotating the analyser through 90°, the tint of the light transmitted changes to its complementary.

**Parallel Rays: Biaxal Crystal.**—Selenite is a crystalline form of calcium sulphate ($CaSO_4 + 2H_2O$). It is a biaxal crystal, which naturally cleaves parallel to the plane containing the optic axes (p. 538). If light is transmitted normally through a crystal of selenite, it is divided into two rays travel-

ling with different velocities, the vibrations in these rays being parallel and perpendicular to the bisector of the angle between the optic axes (p. 544). Since in the case considered, the waves travel along the axis of $y$ (Fig. 279, p. 536), the velocities of the two sets of waves are respectively equal to $a$ and $\gamma$, when $a > \gamma$. Therefore, if $\mu_1$, $\mu_2$, $\mu_3$, are the three principal refractive indices of selenite for a wave-length $\lambda$, where $\mu_1 < \mu_2 < \mu_3$ (p. 537), the difference of phase, $\phi$, between the two sets of waves after travelling a distance $\delta$ along the axis of $y$, is given by

$$\phi = 2\pi \frac{\delta}{\lambda}(\mu_3 - \mu_1).$$

Quarter and half wave plates may accordingly be made from selenite, and the colour phenomena described above may also be produced, the bisector of the angle between the optic axes of the selenite occupying the same position as the single optic axis of the quartz.

Mica is a biaxal crystal which cleaves naturally in planes perpendicular to the bisector of the angle between the optic axes. Accordingly, light transmitted normally through a film of mica traverses the axis of $z$ (Fig. 278, p. 535), and is divided into two coincident rays travelling with the velocities $a$ and $\beta$, where $a > \beta$. Let an imaginary plane be drawn perpendicular to the surface of the mica, so as to contain the optic axes. Then the vibrations of the faster ray are performed in this plane, while those of the slower ray are performed perpendicular to it. Let $\mu_1$, $\mu_2$, and $\mu_3$ be the principal refractive indices of mica for a wave-length $\lambda$, where $\mu_1 < \mu_2 < \mu_3$. Then the difference of phase, $\phi$, between the two sets of waves after travelling a distance $\delta$ along the axis of $z$, is given by—

$$\phi = 2\pi \frac{\delta}{\lambda}(\mu_2 - \mu_1).$$

Quarter and half wave plates may be made from mica, and the colour phenomena described above may be produced. The most brilliant chromatic effects will be obtained when the vibrations in the incident light are inclined at an angle of 45° to the plane containing the optic axes, and the analyser is adjusted to transmit vibrations parallel, or perpendicular, to those of the incident light.

**Chromatic Effects, using a Double-Image Analyser.**—When polarised light is transmitted through a double-image prism, it becomes divided into two divergent rays, in which the vibrations are performed in perpendicular planes (p. 497). If a double-image prism is substituted for the analysing Nicol in the above experiments, two images of the crystal will simultaneously be seen, corresponding to the images seen separately through a Nicol before and after it has been rotated through a right angle. Consequently, the colours of the two images are complementary; the portions of the images which overlap appear white.

**Divergent Rays: Uniaxal Crystal.**—Let AB and CD (Fig. 293) represent opposite parallel faces of a plate of a positive uniaxal crystal cut perpendicular to the optic axis. The plane of the paper is thus a principal plane of the crystal (p. 487.) A ray, whether polarised or not, is transmitted without alteration when incident normally on the plate (the crystal is supposed not to possess the rotary power characteristic of quartz, p. 503). An incident ray inclined to the normal is divided into two rays which travel through the crystal in different directions and with different velocities. Let FG be a polarised ray inclined to the normal GN. Draw the trace GH of the incident wave front. The vibrations in the latter may be resolved into components respectively parallel to GH, and to a line through G at right angles to the plane of the paper. The component vibrations parallel to GH give rise to the extraordinary wave front, while those perpendicular to the plane of the paper give rise to the ordinary wave front. Draw HD parallel to FG. While the incident wave GH is travelling through the distance HD, let the ordinary and extraordinary wavelets, generated at G, respectively spread out into the sphere KLM, and the ellipsoid PLQ. Through D draw the planes DR and DS perpendicular to the plane of the paper, touching the sphere and

FIG. 293.—Rays transmitted by Positive Uniaxal Crystal.

ellipsoid at R and S respectively. Then DR is the ordinary, and DS the extraordinary wave front within the crystal. Also, GR, produced to O, gives the path of the ordinary ray within the crystal, while GS produced to E gives the path of the extraordinary ray. On leaving the crystal both of these rays become parallel, since they were originally derived from a single ray, FG. If the thickness of the crystalline plate is small, the emergent rays will be so close together as to merge into one. The result is therefore practically the same as if the ordinary and extraordinary waves had travelled, with slightly different velocities, along the line GO. The ordinary wave travels with a uniform velocity whatever may be the inclination of its path to the optic axis. On the other hand, the extraordinary wave travels with its maximum velocity (equal to that of the ordinary wave) when its path coincides with the optic axis but its velocity decreases as the inclination of the path to the optic axis increases. Thus, the greater the inclination of the path GO to the optic axis, the greater is the phase difference between the ordinary and extraordinary wave disturbances at O. On emergence, the ordinary and extraordinary wave vibrations combine to produce a resultant vibration which is, in the general case, elliptical; circular and rectilinear vibrations being considered as limiting cases.

Let us now suppose that the incident light consists of a pencil of polarised monochromatic rays converging toward the point G on the lower side of the crystalline plate. The rays within the plate diverge from G, and those transmitted along the optic axis suffer no alteration. Produce the line GL to cut AB in T, and with T as centre draw an imaginary circle on the upper surface of the plate; then points on this circle will be illuminated by rays which make equal angles with the optic axis. Through each point on the circle passes an ordinary, and the corresponding extraordinary ray, derived from a single incident ray; and since the phase difference between the emergent ordinary and extraordinary waves depends only on the inclination of their path to the optic axis, it follows that at all points on the circle the phase difference between the emergent ordinary and extraordinary waves is constant. We may therefore, with T as centre, draw consecutive circles, such that each is characterised by a certain phase difference between the ordinary and extraordinary rays passing through its circumference.

Let AB, CD, and EF (Fig. 294) be quadrants of circles described, with T as centre, on the upper surface of the plate. Let the phase difference between the ordinary and extraordinary rays passing through points on AB be equal to $2\pi$, while the phase difference corresponding to points on CD is equal to $\{2\pi + (\pi/2)\}$, and that corresponding to points on EF is equal to $3\pi$. Let the vibrations in the incident polarised light be parallel to TF. Radial lines drawn from T will indicate the traces of various principal planes of the crystal. The amplitude of the incident polarised light may be represented by a straight line parallel to TF. To find the amplitudes of the ordinary and extraordinary rays leaving the crystal at a point on AB, CD, or EF, draw a rectangle with the incident wave amplitude as diagonal, and two sides parallel to the trace of the principal plane passing through the point; the extraordinary and ordinary wave amplitudes are respectively equal to the sides of this rectangle, parallel and perpendicular to the trace of the principal plane. It is now easy to determine the resultant vibration in an emergent ray. Since the phase difference between the ordinary and extraordinary wave disturbances at points on AB amounts to $2\pi$, which is equivalent to zero phase difference, the resultant vibrations are precisely similar to the incident wave vibrations. At points on CD the two rectangular vibrations, virtually differing in phase by $\pi/2$, combine to produce vibrations which in general are elliptical. At the point D there is no ordinary wave vibration, since the incident wave vibration is parallel to the principal plane, and is therefore transmitted as an extraordinary wave. Thus, at D the wave vibration is similar to that of the incident light. At C only an ordinary wave is transmitted, since there the incident wave vibrations are perpendicular to the principal plane. Thus, at C the wave vibration is similar to that of the incident light. At a point midway between C

FIG. 294.—Forms of Vibration of Emergent Rays.

and D the ordinary and extraordinary wave vibrations are equal, and since their phases differ by $\pi/2$, the emergent light is circularly polarised. The elliptic vibrations at other points on CD are produced in the manner indicated by the construction.

At points on EF the ordinary and extraordinary wave vibrations differ by $3\pi$, which is equivalent to a phase difference of $\pi$. The resultant vibration is rectilinear, inclined to the trace of the principal plane at an angle equal to 180° *minus* the angle of inclination of the incident wave vibrations. At F and E the resultant vibrations are similar to the incident wave vibrations. At a point midway between E and F the resultant vibrations are at right angles to the incident wave vibrations. The resultant vibrations at other points on EF may be understood from the diagram.

Let us now suppose that the emergent light is analysed by a Nicol. If the principal section of the analyser is at right angles to TF (the direction of vibration of the incident light), then the light emerging from points on the straight lines TE and TF will be intercepted, and the field of view will be crossed by two black brushes at right angles to each other (Fig. 295). One brush is parallel to the principal section of the polariser and the other to that of the analyser. The light emerging from the circle AB will also be intercepted, as will that from the circles corresponding to phase differences of $4\pi$, $6\pi$, $8\pi$, . . . &c. The light emerging from the middle point of the arc EF is polarised perpendicular to the incident light, and will therefore be transmitted by the analyser. Thus, the circle EF, and the circles corresponding to phase differences of $4\pi$, $5\pi$, $7\pi$, . . . will be bright. The point T will thus be surrounded by concentric circles, alternately dark and bright (Fig. 295).

If the analyser is now rotated through 90°, so that its principal section is parallel to TF, the field of view will be crossed by two bright brushes at right angles to each other. It is also easily seen that the circles which were previously bright will now be dark, and *vice versâ* (Fig. 295).

If the incident light is white, the field will be crossed by two rectangular brushes, black or white according as the polarising and analysing Nicols are crossed or parallel. The bright rings corresponding to short wave-lengths will possess smaller diameters than those corresponding to longer wave-lengths, so that the resultant rings are brightly coloured.

Rotating either the analyser or the polariser through 90° causes the colour of each ring to change to its complementary. In applying reasoning similar to the above to a negative crystal cut perpendicular to the axis, the only modification required is that the vibrations parallel to the principal plane are transmitted more quickly than those perpendicular

Nicols Crossed. Nicols Parallel.

FIG. 295.—Calcite Rings and Brushes. (From photographs by Mr. W. B. Croft.)

to it. The resulting rings and brushes are similar in both cases.

**Apophyllite Rings.**—Apophyllite is a crystallised double silicate of potassium and calcium, associated with calcium or potassium fluoride. It is remarkable for being positive for one end of the spectrum, negative for the other end, and singly refracting for an intermediate colour, generally yellow. When examined between crossed Nicols with divergent white light, the rings are approximately white and black, a slight trace of green being observed inside each black ring.

**Double Refraction due to Strain.**—Carefully annealed glass possesses identical properties in all directions, and thus does not exhibit any of the characteristics of a doubly refracting substance. If we look at a source of light through crossed Nicols, a sheet of unstrained annealed glass introduced between them leaves the field dark as before. If, however, the glass is strained, either mechanically, by compressing or bending it; or by heating one part and leaving the rest cool, so as to produce unequal expansion; then light is immediately

transmitted, and beautifully coloured curves show the direction of the lines of strain. It is found that when glass is uniformly extended or compressed, it acts like a doubly refracting crystal of which the axis is parallel to the direction of strain. Brewster made artificial crystals by melting together white wax and resin in equal proportions and compressing a small quantity of the cooled mixture between glass plates. The thin film between the plates acted like a uniaxal crystal cut perpendicular to the axis. A glycerine jujube compressed between glass plates acts in a similar manner.

Dr. Kerr introduced two terminals into holes drilled in a slab of glass, and placed the glass between crossed Nicols, so that the line joining the terminals was perpendicular to that drawn through the centres of the Nicols, and at an angle of 45° with the principal sections of the latter. On connecting the terminals to a powerful Wimshurst machine, the field of view immediately became coloured, thus proving that there is a tension in the glass along the lines of electric force. Dr. Kerr also obtained a similar result when the terminals were placed in a similar position within a vessel containing carbon bisulphide.

**Quartz cut Perpendicular to the Axis.**—When polarised light is transmitted along the axis of a crystal of quartz, the plane of polarisation is rotated (p. 503). Consequently, if the incident light vibrations are performed parallel to TF (Fig. 294), the light emerging from the centre of the field near T will in general be characterised by vibrations inclined to TF, and will not be extinguished when the Nicols are crossed. Since the rotation of the plane of polarisation is greater for short than for long waves (p. 504), the centre of the field will in general be coloured. Black brushes at right angles to each other make their appearance near the outer edge of the field (Fig. 296).

FIG. 296.—Rings and Brushes due to Quartz cut Perpendicular to the Axis. (From a photograph by Mr. W. B. Croft.)

Let two plates of quartz, cut perpendicular to the axis, and exactly similar except that one is right- and the other left-handed, be superposed and placed between crossed Nicols, the incident light being convergent; then the rings and brushes seen take the form shown in Fig. 297. The centre of the field is now dark, since the rotations produced by the right- and left-handed plates just neutralise each other. As we proceed outwards, the arms of the black cross become coloured red on one side, and blue on

Nicols Crossed. Nicols Parallel.

FIG. 297.—Airy's Spirals. (From photographs by Mr. W. B. Croft.)

the other, and curve round spirally. The coloured rings take the forms of broken arcs of spirals. These effects are termed Airy's spirals, from their discoverer. On rotating the polariser, or the analyser, through 90°, the bright parts of the field become dark, and *vice versâ*; at the same time the colour at any point of the field changes to its complementary.

**Circularly Polarised Light: Uniaxal Crystal.**—By placing a quarter wave plate (p. 501) between the polarising Nicol and the crystal to be examined, the incident light may be polarised circularly. The principal plane of the quarter wave plate must, of course, be inclined to the principal section of the Nicol at an angle of 45°. Let us suppose that, looking along the direction of transmission of the light, the direction of the circular vibrations is right-handed (p. 243). On looking at the upper surface of the crystalline plate to be examined, the vibrations of the incident light will be left-handed, or executed in a direction opposite to the motion of the hands of a clock. The circular

vibrations, on entering the crystal (which as before is supposed to be cut perpendicular to the axis), are decomposed into two equal rectilinear vibrations, performed parallel and perpendicular to the principal plane of the crystal. These vibrations are respectively transmitted as the extraordinary and ordinary waves. If the crystal is positive (p. 491), the extraordinary wave is retarded behind the ordinary wave. Since, looking down on the crystal, the circular vibrations of the incident light are left-handed, it follows that on entering the crystal the phase of the ordinary wave which emerges at A is behind that of the extraordinary wave by $\pi/2$. If the inner circle (Fig. 298) marks the points on the upper face of the crystal where the emergent extraordinary wave has fallen a quarter wave-length behind the ordinary wave, then the phases of the ordinary and extraordinary wave vibrations are equal along this circle, since the phase difference introduced during transmission just neutralises the original phase difference of the two sets of waves. Accordingly, the vibrations are rectilinear, performed along the straight lines marked in Fig. 298, which are inclined at an angle of 45° to the traces of the principal planes. If the second circle marks the points on the upper face of the crystal where the emergent extraordinary wave has fallen half a wave-length behind the ordinary wave, then at any point on this circle the vibration parallel to the principal plane is behind that perpendicular to the principal plane by $\pi/2$. The emergent waves are here circularly polarised, the direction of vibration being the same as that of the hands of a clock.

FIG. 298.—Circularly Polarised Light, Uniaxal Crystal.

If the third circle marks the points on the upper face of the

crystal where the emergent extraordinary wave has fallen three-quarters of a wave-length behind the ordinary wave, then, at any point on this circle, the vibration parallel to the principal plane is behind that perpendicular to the principal plane by $\pi$. Accordingly, the resultant vibrations here are rectilinear, inclined at an angle of 135° to the principal plane.

Similar reasoning could be extended to other circles surrounding those already considered. At points on the inner and outer circles (Fig. 298), cut by the same principal plane, the vibrations are at right angles to each other. Consequently, if the analyser is arranged so as to intercept the vibrations from a point on the inner circle, those from the corresponding point of the outer circle will be transmitted. Proceeding around either of these circles, the absolute direction of vibration continually changes. If the analyser is arranged to intercept vibrations parallel to AE, then the points D, H, at the middle points of opposite quadrants of the inner circle, will be black, while the points B, F, at the middle points of the remaining quadrants, will be brightly illuminated. Along the lines AE and CG (produced), the illumination is practically uniform. Thus, the bright rings are discontinuous, and appear as if the portions in any two opposite quadrants were contracted, or expanded, with respect to those in the remaining quadrants (Fig. 299). Using white light as an illuminant, the colour of the portion of a ring in one quadrant is complementary to that of the portions in the adjoining quadrants.

FIG. 299.—Dislocated Rings, due to Circularly Polarised and Plane Analysed Light. (From a photograph by Mr. W. B. Croft.)

In constructing Fig. 298, it was supposed that the crystalline plate under examination was positive. If it had been negative, the ordinary wave would have fallen behind the extraordinary wave during transmission, and on emergence along the inner circle the phase of the ordinary wave would have been $\pi$ behind the extraordinary wave. As a consequence, the vibrations along the inner circle would have been at

right angles to those given in the figure, and points from which the vibrations were previously transmitted by the analyser would now be characterised by vibrations which are intercepted, and *vice versâ*. Using white light as an illuminant, the colour of each point in the field changes to its complementary, when a positive plate is interchanged for a negative plate, or *vice versâ*. This gives us a ready means of distinguishing between positive and negative crystals.

**Circularly Polarised and Analysed Light: Uniaxal Crystal.**—The vibrations transmitted normally upward through the centre of Fig. 300 are circular, their direction, looking down on the crystal, being left-handed, or opposite to that of the hands of a clock. Along the circle where the phase retardation of the extraordinary behind the ordinary wave amounts to $\pi$, the vibrations also are circular, but the direction here is right-handed. Along the circle (not shown in the figure) where the phase retardation amounts to $2\pi$, the vibrations are circular and left-handed, and so on. Let us suppose that a quarter wave plate, say of quartz, is placed above the crystal under examination. Each circularly polarised wave on entering the quarter wave plate is decomposed into two plane-polarised waves, and during transmission the extraordinary wave falls a quarter of a wave-length behind the ordinary wave. Let us suppose that the axis of the quarter wave plate is parallel to EA (Fig. 298). Then the circular vibration from the centre of the figure gives rise to an extraordinary vibration, parallel to EA, and an ordinary vibration, perpendicular to EA, on entering the wave plate. Initially the phase of the ordinary vibration is $\pi/2$ behind the extraordinary vibration; but on leaving the quarter wave plate the retardation of the extraordinary wave just compensates this phase difference, and the emergent rectilinear vibrations are in the same phase; accordingly, they give rise to a resultant rectilinear vibration, inclined at an angle of 45° to the direction EA.

The circular vibrations issuing from points on the crystal, where the phase retardation amounts to $2\pi$, $4\pi$, $6\pi$, . . . &c., will also, after traversing the quarter wave plate, give rise to rectilinear vibrations inclined at 45° to the line EA. The vibrations issuing from points where the phase retardation is equal to $\pi$, $3\pi$, $5\pi$, . . . &c., are circular and right-handed, looking down on the crystal. When decomposed into rectilinear vibrations on entering the quarter wave plate, the phase of the extra-

ordinary vibration, parallel to EA, will be $\pi/2$ behind the ordinary vibration, perpendicular to EA. As the extraordinary wave falls a quarter wave-length behind the ordinary wave during transmission through the quarter wave plate, the two waves differ in phase by $\pi$ on emergence, and they consequently combine to form a resultant rectilinear vibration, inclined at 135° to EA.

If we now analyse the light leaving the quarter wave plate by means of a Nicol of which the principal section is inclined at 45° to the line EA, the light from the centre of the crystal, and from the circles where the phase retardation amounts to $2\pi$, $4\pi$, $6\pi$, . . . &c., will be transmitted, whilst that from the circles where the phase retardation amounts to $\pi$, $3\pi$, $5\pi$, . . . &c., will be intercepted. Thus, the field will show a number of rings alternately bright and dark, without the dark brushes produced when the light is plane-polarised and analysed (Fig. 300). On rotating the analysing Nicol through 90°, without moving the quarter wave plate, the rings previously dark become bright, and *vice versâ*. Simultaneously rotating the analysing Nicol and quarter wave plate produces no change.

FIG. 300.—Continuous Rings, due to Circularly Polarised and Circularly Analysed Light. (From a photograph by Mr. W. B. Croft.)

**Divergent Light: Biaxal Crystal.**—The refraction of light by a biaxal crystal has been considered in the preceding chapter. An incident plane wave is decomposed into two plane-polarised waves in which the vibrations are at right angles to each other, and these waves are, in general, transmitted with different velocities. When the direction of transmission coincides with either of two directions in the crystal, termed the optic axes, the velocities of the two waves are equal. Hence, when divergent light is transmitted through a crystal cut perpendicular to the bisector of the angle between the optic axes, no phase change is introduced between the waves travelling along the optic axes. For other directions, the phase change introduced between the

polarised waves varies. We may draw a number of curves around the points on the upper surface of the crystal cut by the optic axes, such that the phase difference between the polarised waves emerging along each curve is constant. The curves, immediately surrounding the end of either optic axis, are oval ; but those corresponding to greater phase retardations are drawn out so that corresponding curves, surrounding the ends of the two axes, tend to meet ; and one curve generally takes the form of the figure 8, the two loops surrounding the two ends of the axes. Curves corresponding to greater phase retardations approximately take the form of ellipses surrounding the ends of both axes (Fig. 301). The form of these curves can be understood, in a general manner, by reference to the drawing of the biaxal wave surface (Fig. 282, p. 538).

FIG. 301.—Curves of Equal Wave Retardation, Biaxal Crystal.

The directions of vibration in the component polarised waves emerging at any point M (Fig. 301) are determined as follows. From M draw straight lines MF, MF′, to the ends of the optic axes. Then the vibrations are respectively parallel and perpendicular to the line bisecting the angle FMF′ (compare p. 545). Thus, to obtain the component vibrations at M, we must resolve the incident wave vibrations parallel and perpendicular to the bisector of the angle FMF′.

Let us now suppose that the incident light vibrations are parallel to the plane containing the optic axes, or in the direction BOA. Then along the line BOA the only vibrations emerging from the crystal are in the direction BOA. Along the line DOC the only vibrations emerging from the crystal will be perpendicular to DOC, or parallel to BOA. Thus, if the analysing Nicol intercepts vibrations parallel to BOA (*i.e.* if the polariser and analyser are crossed), there will be a black cross in the field, one arm being along BOA, and the other

along DOC (Fig. 302). The coloured rings formed in the remainder of the field can be explained in a manner essentially similar to that described with respect to uniaxal crystals (p. 556). If the analyser is rotated through 90°, the black cross is replaced by a white cross, and the colour of each ring changes to its complementary. When the Nicols are crossed, and the plane containing the axes of the biaxal crystal makes an angle of 45° with the principal section of the polariser (*i.e.* the incident light vibrations make an angle of 45° with the line BOA, Fig. 301), the black brushes take the forms of hyperbolic curves, one passing through the end of each optic axis (Fig. 302).

**Fig. 302.—Rings and Brushes due to Nitre. (From a photograph by Mr. W. B. Croft.)**

When a biaxal crystal, cut perpendicular to one of the optic axes, is placed between crossed Nicols and examined by divergent light, coloured rings similar to those due to a uniaxal crystal are produced. There is, however, only a *single black brush* crossing the field; this brush corresponds to the single brush crossing each of the "eyes" in Figs. 302 and 303.

**Fig. 303.—Rings and Brushes due to Nitre.**

**Apparatus.**—The rings and brushes due to uniaxal crystals, and some biaxal crystals, can be observed by placing the crystal, cut in a suitable direction, between crossed tourmalines, and looking through the combination at a bright cloud. For this purpose tourmalines may be conveniently mounted on wire supports (Fig. 304), in which form they are termed *tourmaline forceps.* Each tourmaline can be rotated independently. To observe the black brushes, rotate one tourmaline till it intercepts the light trans-

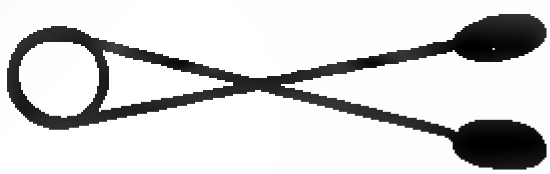

**Fig. 304.—Tourmaline Forceps.**

mitted by the other, and then insert the crystal. Ferrocyanide of potassium is a uniaxal crystal crystallising in tablets which cleave perpendicular to the optic axis, so that a crystal can, by trial, easily be split down to a requisite thickness, and no grinding or polishing is needed. Mica is a biaxal crystal which cleaves perpendicular to the bisector of the angle between the optic axes, and may be used to exhibit the rings and brushes of biaxal crystals. In different samples of mica the angle between the optic axes varies considerably ; some specimens of mica act almost as uniaxal crystals. When the angle between the axes is great, the "eyes" can only be seen separately by looking obliquely through the tourmalines with the mica between them.

Fig. 305 represents a more elaborate piece of apparatus designed by Mr. Lewis Wright,[1] for observing and projecting polarisation effects.

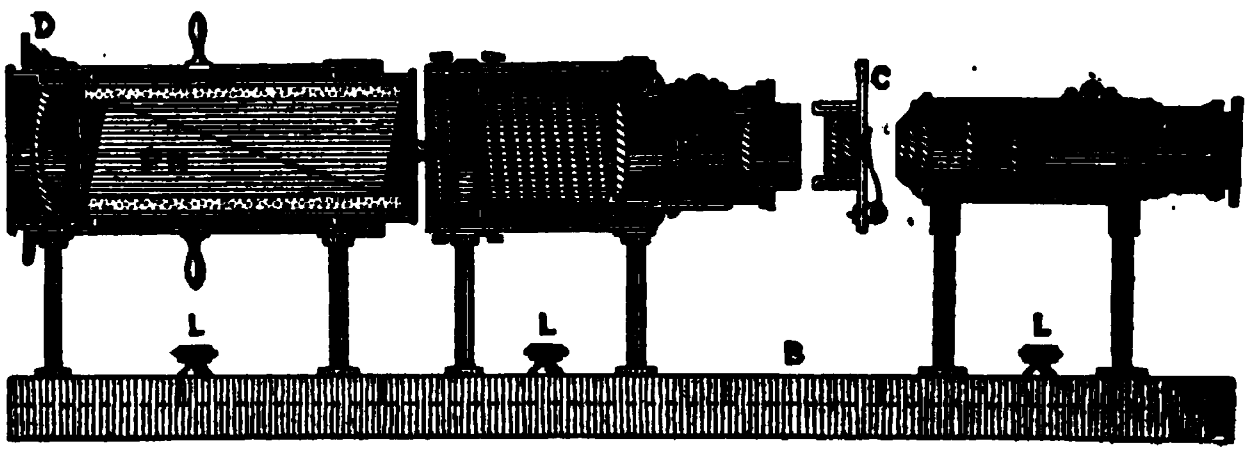


FIG. 305.—Apparatus for projecting Rings and Brushes.

PN is the polarising Nicol ; after traversing this, light is converged by the lens system F, and is finally brought to a focus on the crystal by the lenses mounted on the slide-holder C. The latter fits on the nozzle of F, and the crystal is held by the small spring shown. The light then traverses a series of lenses forming an objective, and an image of the rings and brushes is formed at the second principal focus of this objective ; this is due to the circumstance that the interfering rays O and E (Fig. 293, p. 555) are parallel, and are brought to a focus in the second focal plane of the objective. The lenses H and K focus the rings on a screen, the rays crossing each other in the analysing Nicol AN.

**Determination of the Angle between the Axes of a Biaxal Crystal.**—Fig. 306 represents a piece of apparatus which may

[1] *Light*, Lewis Wright, p. 249 (Macmillan).

be used for this purpose. Light is polarised by reflection from the sheet of black glass AA (or the glass may be removed, and a Nicol used as polariser), and is then focussed, by a lens B, on the crystal supported in the slide-holder K. Another lens, C, forms an image of the rings and brushes at F, which can be viewed through an eye-piece lens, D, and an analysing Nicol, T. The biaxal crystal is mounted on K, so that the two "eyes" (Figs. 302 and 303) appear in the same vertical line. The slide-holder

Fig. 306.—Apparatus for determining the Angle between the Axes of a Crystal.

is then rotated about a horizontal axis till one "eye" appears in the middle of the field of view, and the position of the vernier N is noted. The slide-holder is then rotated till the other "eye" occupies the centre of the field of view, and the position of the vernier is again observed. The difference between the two vernier readings gives the "apparent" angle between the optic axes. If the crystal is surrounded with a liquid in which the velocity of light is equal to the wave velocity along the optic axes of the crystal, the true angle between the optic axes may be directly observed.

**Dispersion of the Optic Axes.**—In the majority of biaxal

crystals the optic axes have different directions for different wave-lengths of light. In brookite and chrysoberyl the optic axes, for rays from opposite ends of the spectrum, lie in planes at right angles to each other. If the system of rings produced by these crystals are examined by monochromatic light, as the wave-length of the light is continuously varied the "eyes" draw nearer to each other, until for a certain wave-length the rings and brushes resemble those of a uniaxal crystal. On still further varying the wave-length, the "eyes" separate in a direction at right angles to that first observed.

**Mitscherlich's Experiment.**—In most biaxal crystals the inclination of the optic axes varies with the temperature. The rings and brushes of selenite undergo an interesting change as the temperature is raised. At first the eyes draw nearer to each other, until they coalesce into a single set of rings, similar to those characteristic of a uniaxal crystal. On raising the temperature still farther, the eyes separate in a direction at right angles to that first observed. On cooling, the axes generally return to their original directions. On cooling after long-continued heating, however, the crystal may return only to the uniaxal stage. Accordingly, when selenite crystals, possessing the properties of uniaxal crystals, are found in rocks, we may infer that these rocks have suffered prolonged heating at some previous time.

## Questions on Chapter XX

1. Explain the coloration produced in parallel light by thin crystalline plates placed between the polariser and the analyser of a polariscope.

2. Give a general explanation of the optical phenomena displayed by a thin plate of a uniaxal crystal, cut perpendicular to the axis when viewed in convergent light between crossed Nicol's prisms.

3. Describe, in a general manner, the formation of the rings and brushes seen when convergent light, traversing a plate of crystal, is viewed between crossed Nicols.

Draw a careful diagram of the path of the rays through the lenses of a polariscope arranged to show this.

4. Describe and explain the appearance seen when a thin parallel slice of quartz, cut so that the crystalline axis is normal to the surface, is

viewed between crossed tourmalines held close to the eye, (*a*) by white light, (*b*) by the light of a sodium flame.

5. A plate of uniaxal crystal, cut with the faces perpendicular to the axis, is placed between a polariser and an analyser. How would you arrange a source of light and lenses to show a system of rings on a screen?

Explain how the rings are formed when the polariser and analyser are crossed.

6. How thick should a quarter wave plate of selenite be if cleaved parallel to the plane containing the optic axis? The principal refractive indices of selenite may be taken as 1·530, 1·523, 1·521, for light of wave-length ·00006 cm.

7. Give an account of experiments which have been made on the effect of electric stress, upon a beam of polarised light traversing the dielectric between two conductors at different potentials.

## PRACTICAL

1. Arrange the polariscope so as to exhibit the characteristic rings and brushes of the specimens of nitre and calc-spar given you.

2. With the mica supplied, construct a quarter wave plate for sodium light.

3. Examine by means of a tourmaline polariscope the crystals supplied to you; describe what you see, and state the inferences you draw.

4. A specimen of a crystal placed in convergent light between crossed Nicols is exhibited to you under such conditions that its temperature can be altered, the coloured bands produced being projected on a screen. Describe the phenomena you observe, before, during, and after the heating of the crystal, and state what conclusions you draw from them.

# ANSWERS TO QUESTIONS

## Chapter I (p. 20)

2. When the screen is close to the mirror, the illuminated patch will be rectangular; when it is at some distance from the mirror, an image of the sun will be formed. The effect is very much the same as if the sun's rays were transmitted obliquely through a small square aperture. (See p. 5.)

## Chapter III (pp. 80–82)

2. Raised by $\frac{1}{3}$ of its true distance below the surface.
9. Angles of prism, 60°, 60°, and 60°.
10. $f_2 = -32{\cdot}2$ mm.
13. $v = -4{\cdot}8$ ins.; length of image = 3 ins.
14. (*a*) $v = -20{\cdot}7$ cms.; length of image = $0{\cdot}345$ cm.
    (*b*) Image is depressed by $3{\cdot}4$ cms.
15. Focal length in water : focal length in air = 4 : 1.
17. (*a*) See also p. 446.
19. (*a*) + 16 ins. from concave lens. (*b*) + 90 ins. from convex lens.
20. The second principal focus of the lens facing the incident rays must coincide with the first principal focus of the other lens.

## Chapter IV (pp. 107–109)

2. $\mu = \sqrt{2}$.
3. The limiting angle of the prism is such that the ray is incident internally on the second face at the critical angle. Thus—

$$\alpha = \sin^{-1}\left(\frac{\sin i_1}{\mu}\right) + \sin^{-1}\left(\frac{1}{\mu}\right).$$

6. For (2), see p. 210.
10. $r_1' = -r_2' = 30$ cms.; $F = -18{\cdot}1$ cms.
13. $f' = 78{\cdot}9$ cms.; $F = -136{\cdot}5$ cms.
15. $1^\circ{\cdot}5$.

## Chapter VII (pp. 157–158)

6. There is no such point, since in general the principal points of the lens combination do not coincide. (Compare Question 10.)

9. 3·94 sq. mm.

10. $F = \dfrac{f_1 f_2}{f_1 + f_2 + d}$; $\alpha = -\dfrac{f_1 d}{f_1 + f_2 + d}$; $\beta = \dfrac{f_2 d}{f_1 + f_2 + d}$.

## Chapter IX (p. 198)

1. (*a*) − 5 dioptres; (*b*) + 20 cms.
2. 20/6.
3. 11.

## Chapter X (p. 218)

1. 0·55 inch beyond objective.

3. If the telescope is adjusted for the normal eye at rest, magnification = 23.

6. If the telescope is adjusted for the normal eye at rest, focal length of eye-piece = 0·72 inch nearly.

7. If distance of distinct vision = 10 ins., and microscope is adjusted to give maximum magnification, object must be placed 1·14 ins. (nearly) beyond objective.

## Chapter XIII (pp. 328–329)

5. See also p. 446.

## Chapter XVI (pp. 425–426)

7. Use formulæ—

$$2\mu\delta \cos r = n\lambda.$$

$$\delta = D^2/8R.$$

$$R = \frac{\sqrt{3} \times 1 \times 10^7}{8 \times 3 \times 589} = 1{,}226 \text{ cms.}$$

## Chapter XVII (pp. 469-470)

5. See p. 446. Points remote from the axis, with respect to which the convergent wave surface comprises any odd number of half-period elements, will be bright. (Compare p. 439.)

## Chapter XVIII (pp. 509–511)

18. 0·00162 cm.

## Chapter XX (pp. 570–571)

$$\delta = \frac{\cdot 00006}{4 \times (1\cdot 530 - 1\cdot 521)} = 0\cdot 0016 \text{ cm.}$$

# INDEX

(*Names of persons are printed in italics.*)

THE END

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